Multi-meson systems from Lattice QCD

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November 6, 2012

Outline

- High density systems exist in the evolution of the early universe, and it may exist in the core of a neutron star.
- Such system can also be produced experimentally, for example in LHC, where high temperature and hight density systems may be produced.
- Famous sign problem for non-zero baryon chemical potential systems makes the simulation at high densities exponentially expensive, however non-zero isospin chemical potential does not have the sign problem.
- Study multi-meson system is the first step toward more complicated and more interesting multi-baryon system.

Constructing a n-π^+ system

The correlation functions for a system of $\overline{n} = \sum_{i=1}^{N} n_i$ π^+'s with n_i π^+s from the i^{th} source is defined as:

$$C_{n_1,\ ...,\ n_M}(t) = \left\langle \left(\sum_{\mathbf{x}} \pi^+(\mathbf{x}, t) \right)^{\overline{n}} \left(\pi^-(\mathbf{y_1}, 0) \right)^{n_1} ... \left(\pi^-(\mathbf{y_N}, 0) \right)^{n_N} \right\rangle , \quad (1)$$

where $\pi^+(\mathbf{x}, t) = \overline{d}(\mathbf{x}, t)\gamma_5 u(\mathbf{x}, t)$ and $\pi^-(\mathbf{x}, t) = \overline{u}(\mathbf{x}, t)\gamma_5 d(\mathbf{x}, t)$.

$C_n(t)$

- Number of contractions $\propto \overline{n}!$.

• Without thermal contributions

$$C_{n\pi^+}(t) \quad = \quad Z_0 e^{-(E_n)T/2} \cosh(E_n \cdot (t - T/2)) + \cdots$$

where the ellipsis denote contributions from excited states.

• With thermal contributions

$$C_{n\pi^+}(t) \quad = \quad \sum_{m=0}^{n} \binom{n}{m} Z_m^n e^{-(E_m + E_{n-m})T/2} \cosh((E_m - E_{n-m}) \cdot (t - T/2)$$

Additional terms are thermal contributions.

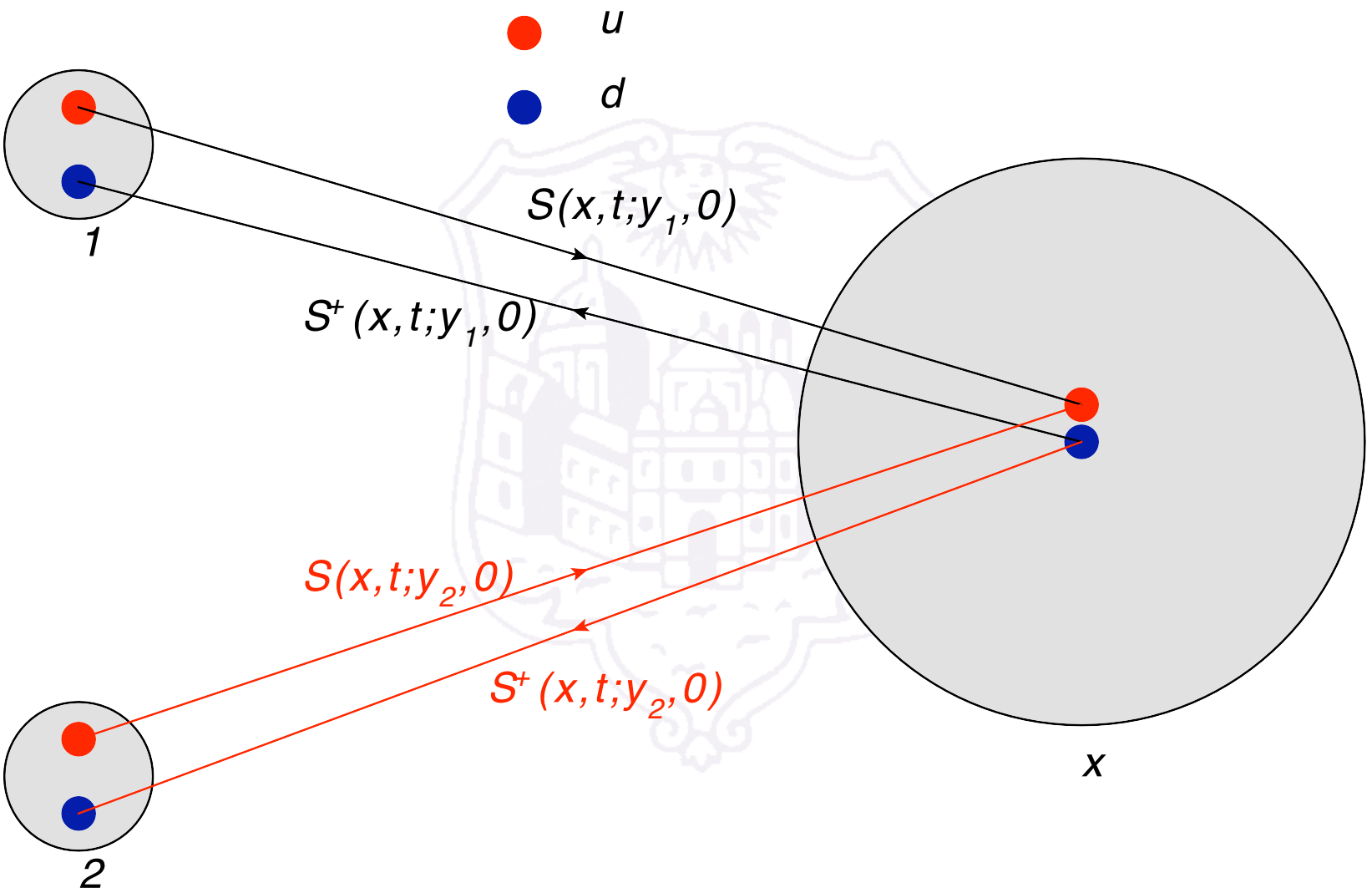
First attack on the $n!$ contractions

The correlator $C_{n_1,\ldots,n_N}$ can be identified as the term with prefactor $\prod_{i=1}^{N} \lambda_i^{n_i}$ from the expansion of $\det[1 + \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_N P_N]$, where N is the number of sources, and the $12N \times 12N$ matrices P_k are given by:

$$P_k = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline P_{k,1} & P_{k,2} & \ldots & P_{k,N} \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \tag{2}$$

with 12×12 sub-blocks

$$P_{k,i}(t) = \sum_{\mathbf{x}} S(\mathbf{x}, t; \mathbf{y}_i, 0) S^{\dagger}(\mathbf{x}, t; \mathbf{y}_k, 0), \tag{3}$$


u
d
1
$S(x,t;y_1,0)$
$S^+(x,t;y_1,0)$
$S(x,t;y_2,0)$
$S^+(x,t;y_2,0)$
2
x

Recursion relationship of $C_n(t)$ from 1 source

$$C_{n\pi^+}(t) = \left\langle \left(\sum_{\mathbf{x}} \pi^+(\mathbf{x}, t) \right)^n \left(\pi^-(\mathbf{0}, 0) \right)^n \right\rangle$$

$$\begin{aligned} C_n(t) &= (-1)^n \, n! \, \langle \, R_n \, (\, t \,) \, \rangle \\ R_{n+1}(t) &= \langle \, R_n \, \rangle \, A - n \, R_n \, A \end{aligned}$$

where

$$A_{ij}(t) = \sum_{\mathbf{x}} \left[S(\mathbf{x}, t; \mathbf{0}, 0) \right]_{ik} \left[S^\dagger(\mathbf{x}, t; \mathbf{0}, 0) \right]_{kj}$$

$$R_1 = A, \ \langle R_1 \rangle = \langle A \rangle$$

A is a 12×12 matrix, and $\langle \quad \rangle$ is to take the trace of a matrix.

$C_{n\pi}$ from 2 sources

- Maximal number of pions can be put in a single source is $N_c N_s = 12$. Two sources are required for study of more than 12 pions.

Correlation function of a system putting $n_1\pi^+$ in one source and $n_2\pi^+$ in another source is defined as:

$$C_{(n_1\pi_1^+\ ,\ n_2\pi_2^+)}(t) = \left\langle \left(\sum_{\mathbf{x}} \pi^+(\mathbf{x}, t) \right)^{n_1+n_2} \left(\pi^-(\mathbf{y_1}, 0) \right)^{n_1} \left(\pi^-(\mathbf{y_2}, 0) \right)^{n_2} \right\rangle$$

Recursion relation for 2 sources

$$C_{(n_1\pi_1^+\ ,\ n_2\pi_2^+)}(t) \;=\; (-)^{\overline{n}}\ \frac{\overline{n}!}{\overline{n}C_{n_1}}\ \langle\ Q_{(n_1,n_2)}\ \rangle$$

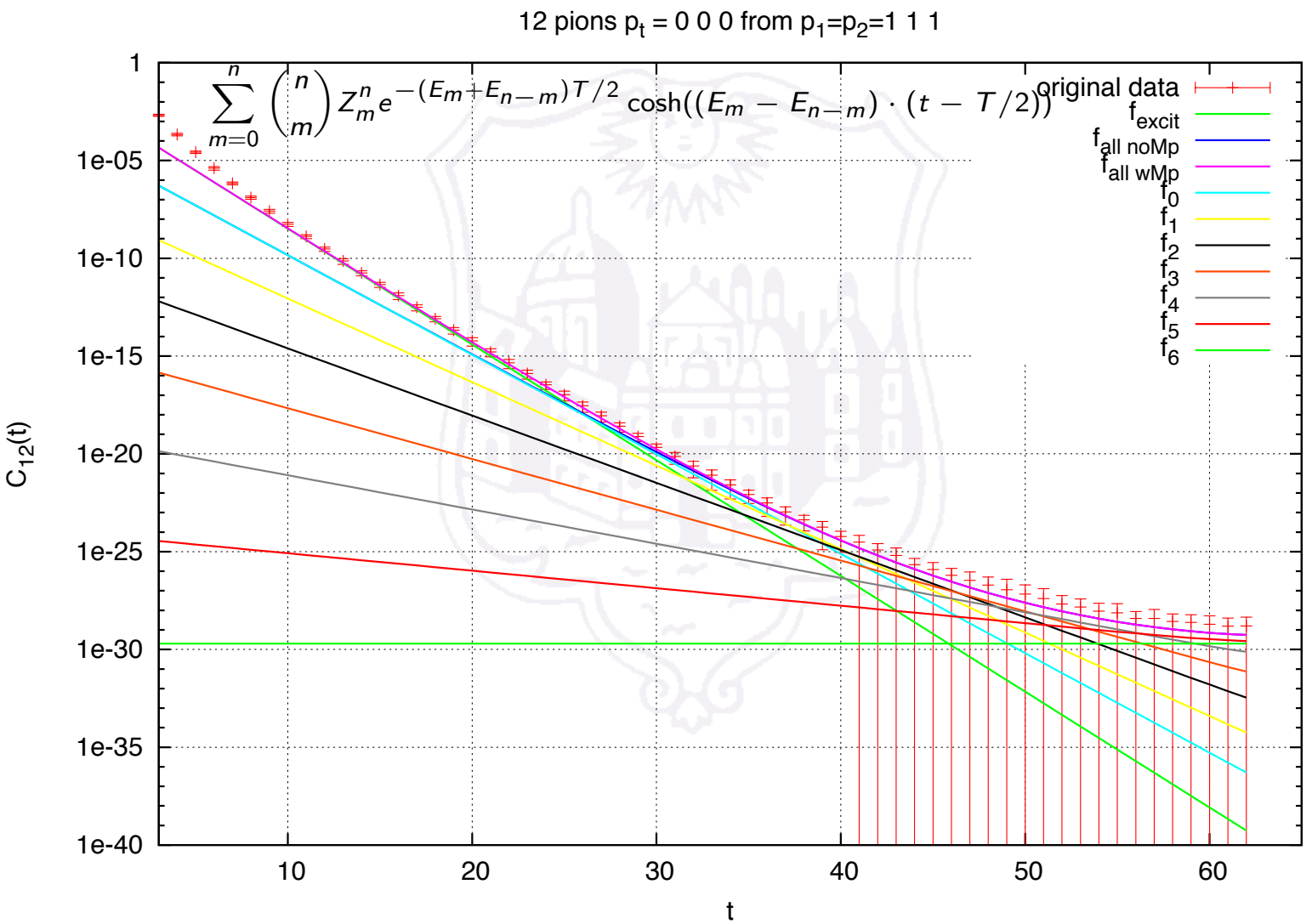
$$\begin{aligned} Q_{(n_1+1,n_2)} \;&=\; \langle\ Q_{(n_1,n_2)}\ \rangle\ P_1 \;-\; (n_1+n_2)\ Q_{(n_1,n_2)}\ \ P_1 \\ &+\; \langle\ Q_{(n_1+1,n_2-1)}\ \rangle\ P_2 \;-\; (n_1+n_2)\ Q_{(n_1+1,n_2-1)}\ \ P_2 \end{aligned}$$

where $\overline{n} = n_1 + n_2$, and Q's,P's are all 24×24 matrices.

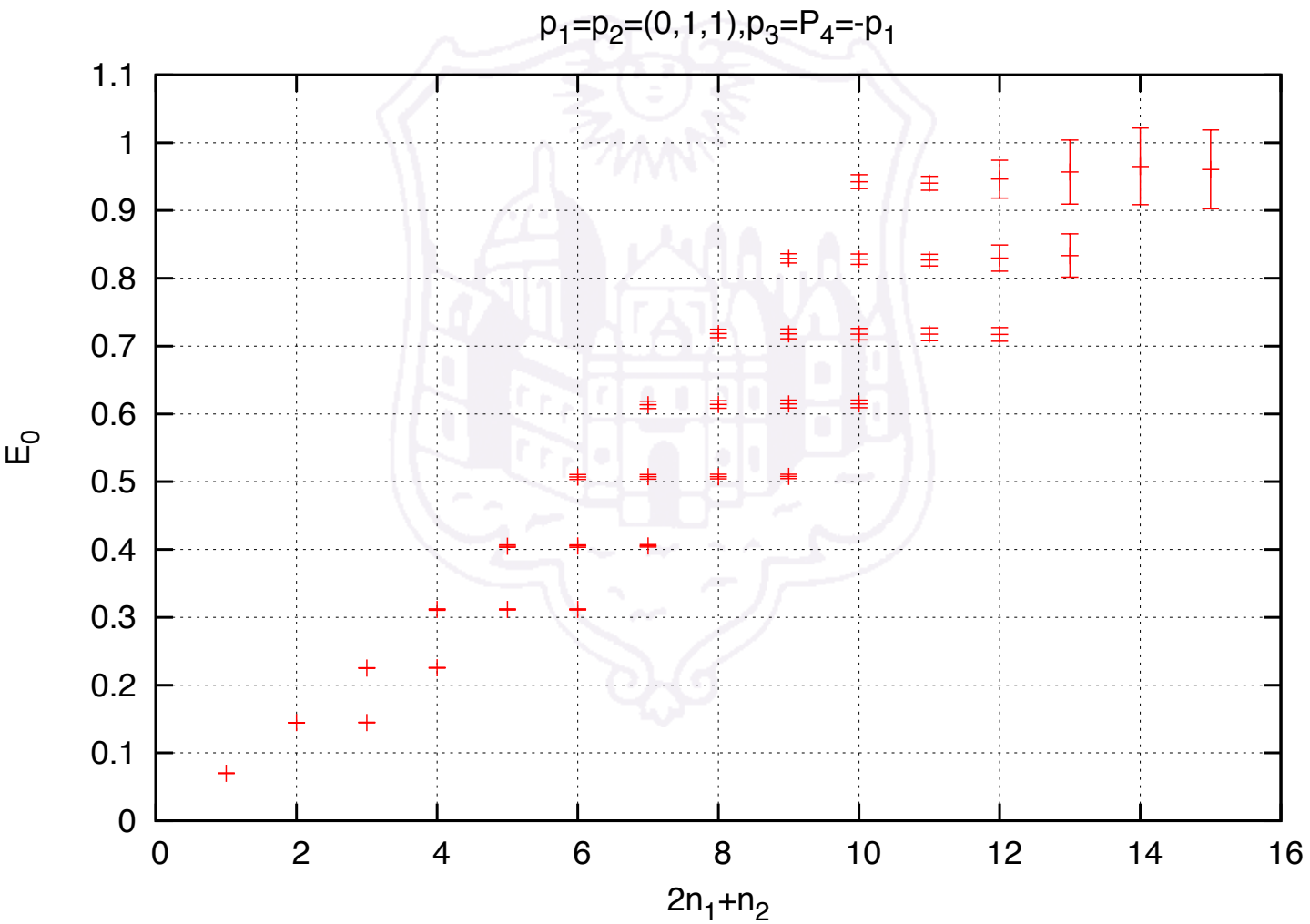
- All combinations of of n_1 and n_2 are required for $n_1 + n_2 = n$ in order to compute $C_{n_1',n_2'}$ for $n_1' + n_2' = n + 1$.

n(# of sources)	1	2	3	4	5
Max # of pions	12	24	36	48	60
total # of combinations	12	167	2195	28559	371291

Extract ground state energies, $E_{n\pi}$



E_{n_1,n_2} for $n_1 + n_2 = n$



Another attempt to study n-π system

- Ground state energies are independent of distribution of π^+.
- We can also identify a combined correlator $C_{\bar{n}\pi}(t)$ as the term having prefactor λ^n from the expansion of $\det[1+\lambda A]$, with

$$A = P_1 + P_2 + \ldots + P_N = \left(\begin{array}{c|c|c|c} P_{1,1} & P_{1,2} & \ldots & P_{1,N} \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline P_{k,1} & P_{k,2} & \ldots & P_{k,N} \\ \hline \vdots & \ldots & \ldots & \ldots \\ \hline P_{N,1} & P_{N,2} & \ldots & P_{N,N} \end{array}\right). \qquad (4)$$

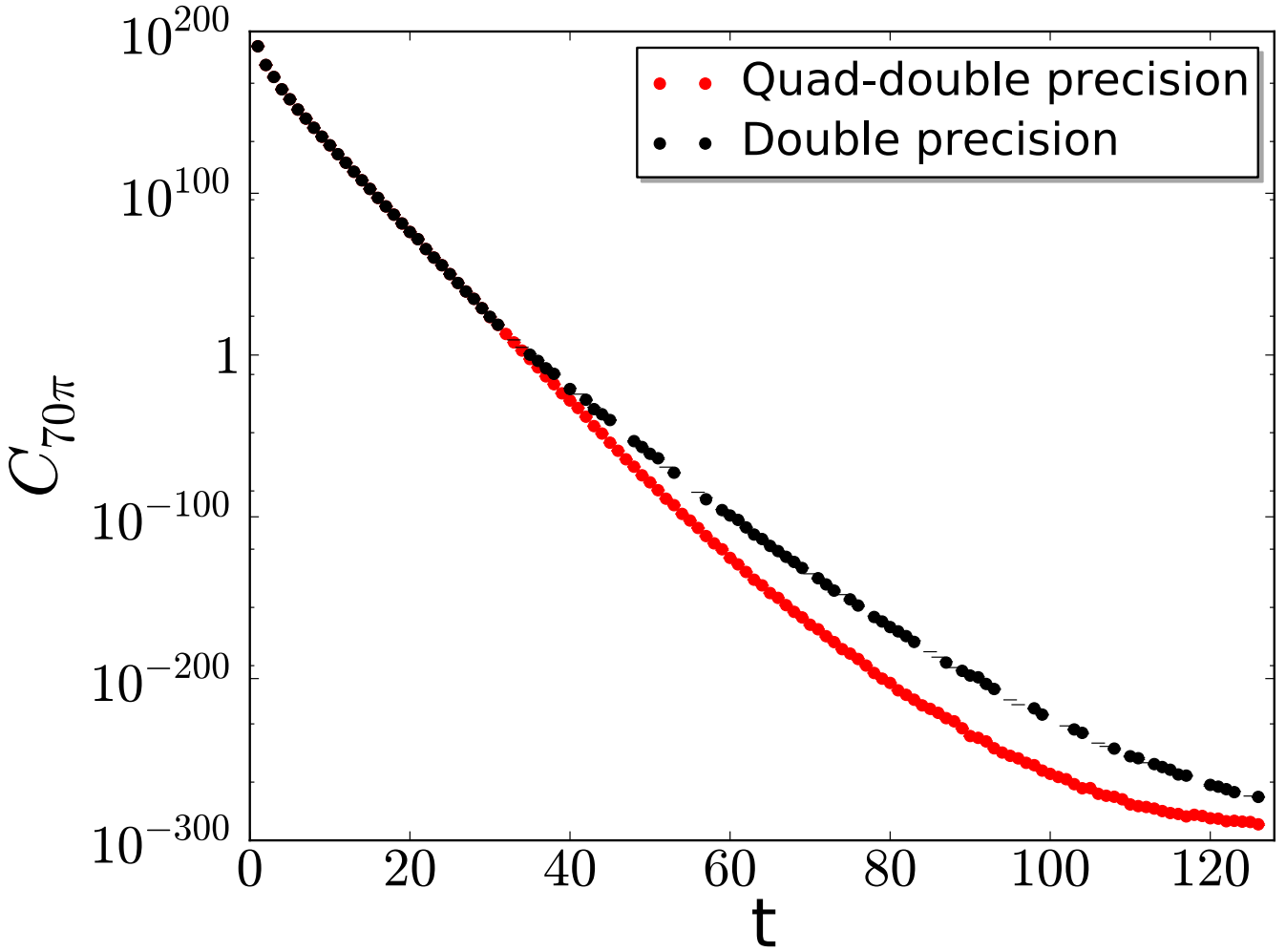
- $\det[1+\lambda A] = 1 + \lambda C_{1\pi} + \lambda^2 C_{2\pi} + \ldots + \lambda^{12N} C_{12N\pi}$.

$$\det[1+\lambda A] = 1 + \lambda C_{1\pi} + \lambda^2 C_{2\pi} + \ldots + \lambda^{12N} C_{12N\pi}.$$

- 1.Compute $\det[1+\lambda A]$ for $12N$ different λ's, and solve a linear equation,
- 2.Set $\lambda = exp(i2\pi f_0 t')$, and identify C_n as the magnitude of frequency nf_0,
- 3.Identify odd/even property of λ^n,
- 4.Identify $C_{n\pi}$ with eigenvalues of the matrix A.

• Computational cost for the recursion relation is $N^4 \exp(2.8(N-1))$. • Computational cost for above methods are $(N^3 \sim N^4)$. • The recursion relation method, and first three above methods can be extended to study systems of mixed species. Method 4 is only applicable for one species systems.

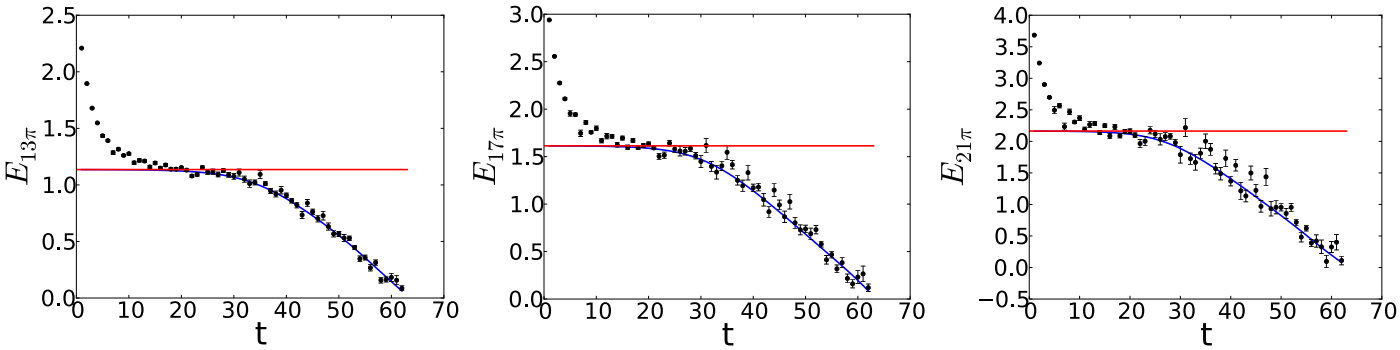
$C_{70\pi}(t)$


Quad-double precision
Double precision
10^{200}
10^{100}
1
10^{-100}
10^{-200}
10^{-300}
$C_{70\pi}$
0
20
40
60
80
100
120
t

Numerical simulations

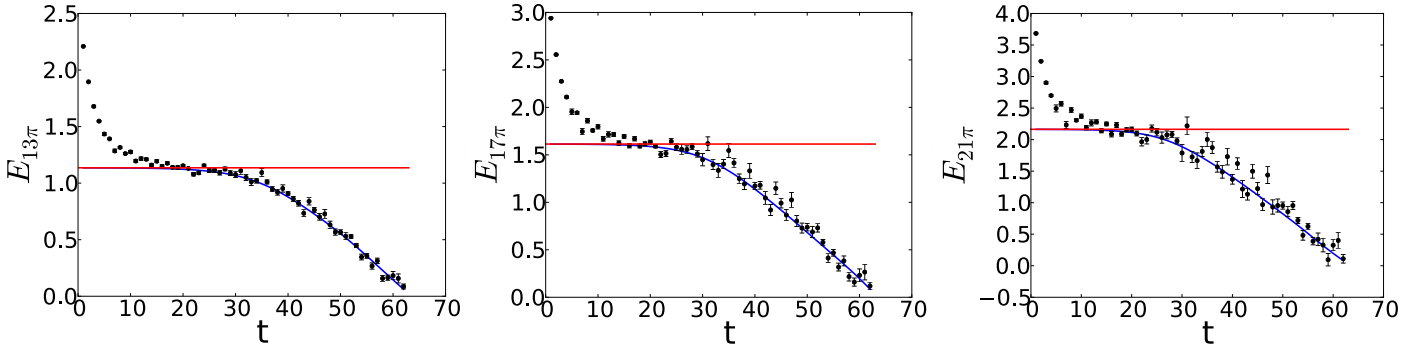
- Simulations are performed on three anisotropic ensembles, $L^3 \times T = \{16^3 \times 128, 20^3 \times 256, 24^3 \times 128\}$, with the anisotropic parameter $\xi = a_s/a_t \approx 3.5$, where $a_s(a_t)$ is the spatial(temporal) lattice spacing and $a_s = 0.125$ fm.

$$20^3 \times 128$$



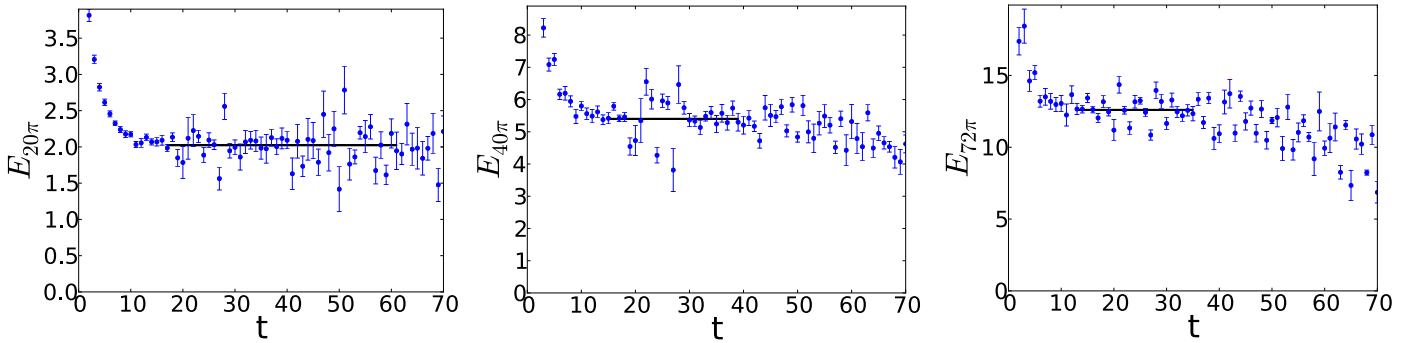
- $A \pm P$ method has been used on $T = 128$ ensembles to double the temporal extent.

Figure : $20^3 \times 128$



- Extremely long plateau region in the effective mass plot enables us to fit a single exponential to extract the ground state energies, $E_{n\pi^+}$, for n up to 72 from all three ensembles.

Figure : $20^3 \times 256$



$E_{n\pi}$ and isospin chemical potential $\mu_I = \frac{dE}{dn} \approx \frac{E_n - E_{n-1}}{1}$

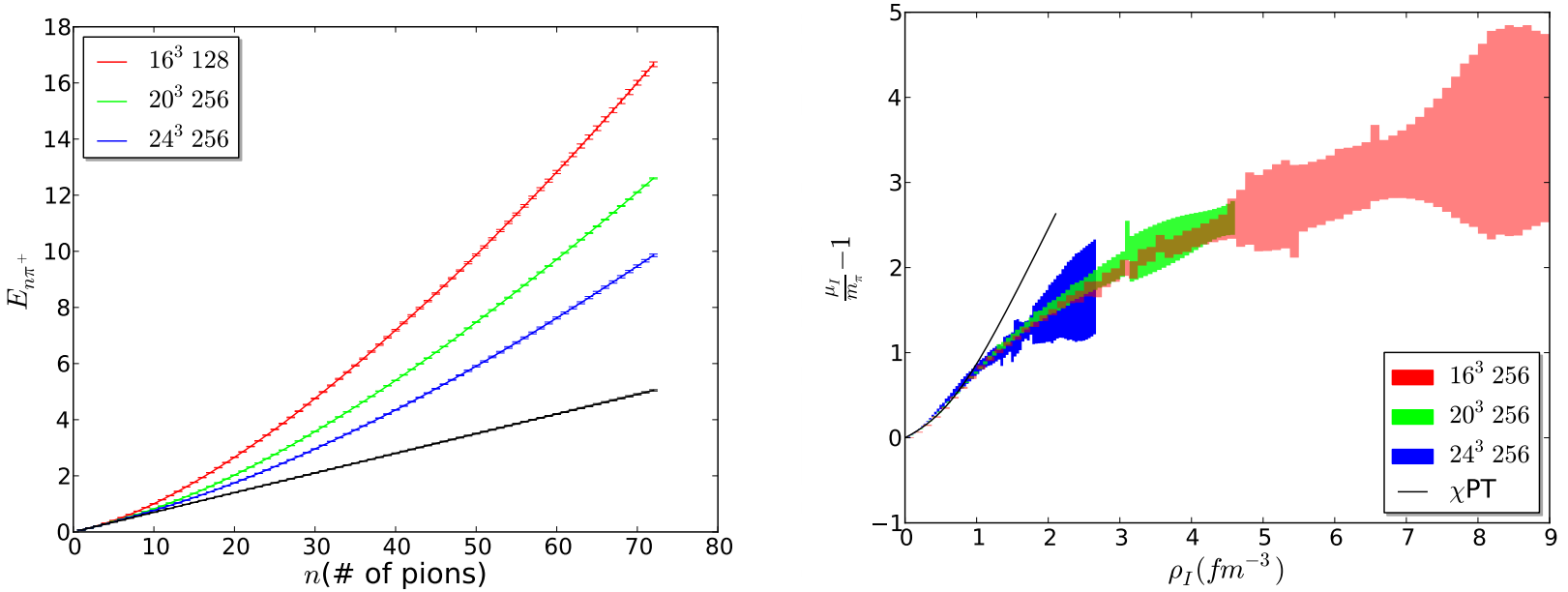
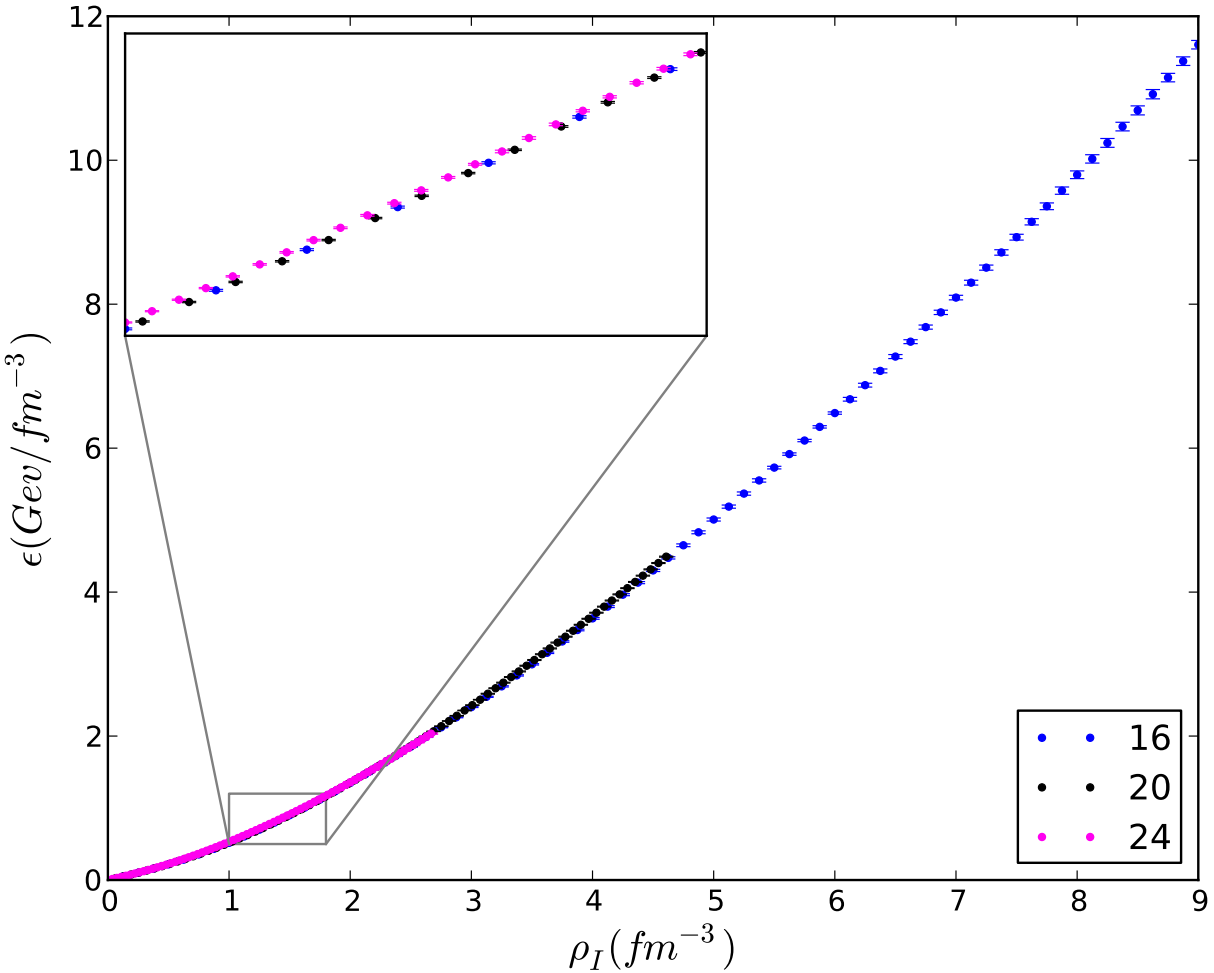


Figure : The left panel shows energies of a rest system of n-π^+($E_{n\pi^+}$), the right panel shows the isospin chemical potential ($\mu_I(n)$) as a functions of isospin density. The solid black line is from expectations of χPT.

Energy density $\epsilon_n = \frac{E_n}{V}$



Lüscher's method

Scattering length,a, and phase shift, $\delta(p)$, can be calculated from the energy shifts of two particle states in a finite volume, $\Delta E \equiv E_2 - 2E_1 = 2\sqrt{\mathbf{p}^2 + m_\pi^2} - 2E_1$ by using:

$$p \cot \delta(p) = \frac{1}{\pi L} \mathbf{S}\left(\left(\frac{pL}{2\pi}\right)^2\right) \quad . \tag{5}$$

The regulated three-dimensional sum, $\mathbf{S}(x)$, is

$$\mathbf{S}(x) \equiv \lim^{\Lambda\to\infty} \left(\sum_{\mathbf{j}}^{|\mathbf{j}|<\Lambda} \frac{1}{|\mathbf{j}|^2 - x} - 4\pi\Lambda \right) \quad , \tag{6}$$

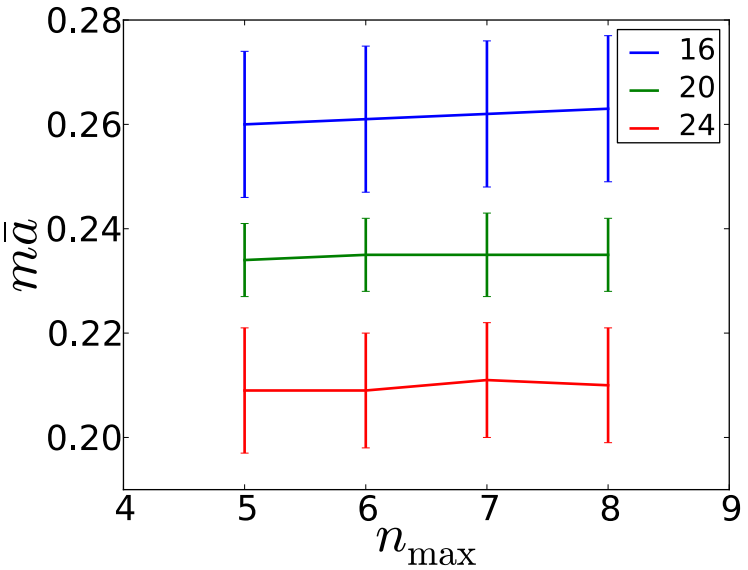
where the summation is over all triplets of integers $\mathbf{j}$ such that $|\mathbf{j}| < \Lambda$.

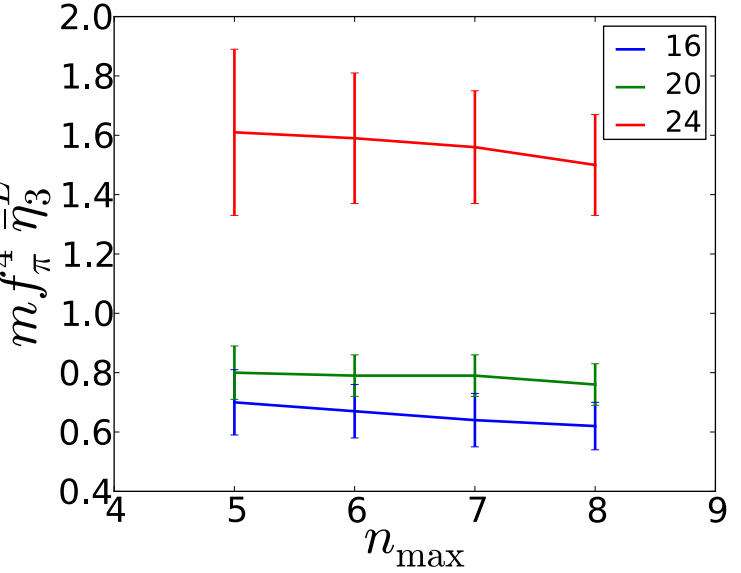
Finite volume expansion

By performing an expansion in small $1/L$, the energy shift of n identical bosons in a finite volume, $\Delta E_n = E_n - nE_1$, is given by:

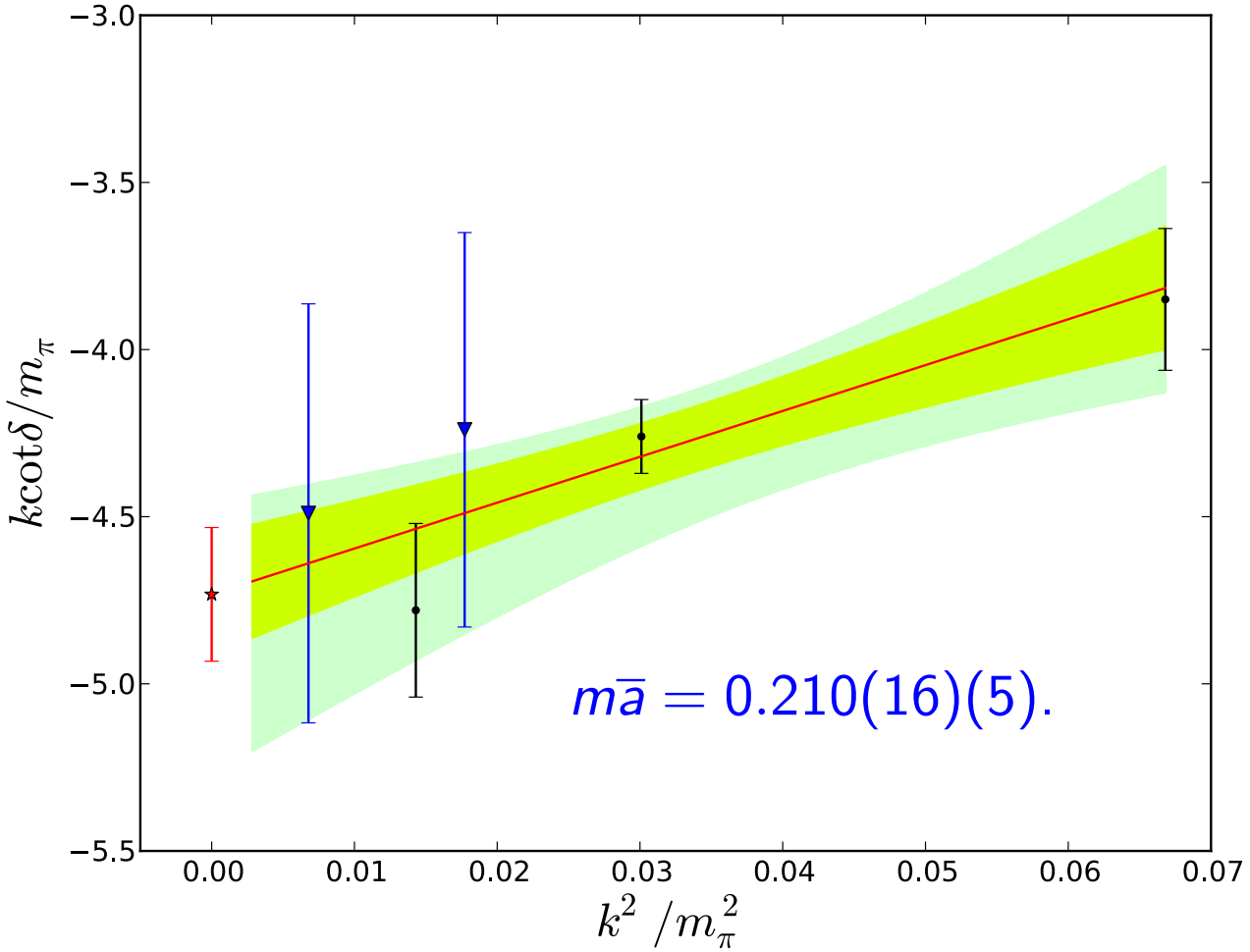
$$
\begin{aligned}
\Delta E_n \quad = \quad & \frac{4\pi\,\bar{a}}{M\,L^3}\binom{n}{2}\Bigg\{1 - \left(\frac{\bar{a}}{\pi\,L}\right)\mathcal{I} + \left(\frac{\bar{a}}{\pi\,L}\right)^2\left[\mathcal{I}^2 + (2n-5)\mathcal{J}\right] \\
& - \left(\frac{\bar{a}}{\pi\,L}\right)^3\left[\mathcal{I}^3 + (2n-7)\mathcal{I}\mathcal{J} + \left(5n^2 - 41n + 63\right)\mathcal{K}\right] \\
& + \left(\frac{\bar{a}}{\pi\,L}\right)^4\Big[\mathcal{I}^4 - 6\mathcal{I}^2\mathcal{J} + (4 + n - n^2)\mathcal{J}^2 + 4(27 - 15n + n^2)\mathcal{I}\,\mathcal{K} \\
& \qquad + (14n^3 - 227n^2 + 919n - 1043)\mathcal{L}\Big]\Bigg\} \\
& + \binom{n}{3}\left[\frac{192\,\bar{a}^5}{M\pi^3 L^7}\left(\mathcal{T}_0 + \mathcal{T}_1\,n\right) + \frac{6\pi\bar{a}^3}{M^3 L^7}\,(n+3)\,\mathcal{I}\right] \\
& + n3\,\frac{1}{L^6}\,\bar{\bar{\eta}}_3^L \quad + \mathcal{O}\left(L^{-8}\right) \quad . \qquad (7)
\end{aligned}
$$

$\bar{a}$ and $\bar{\eta}$



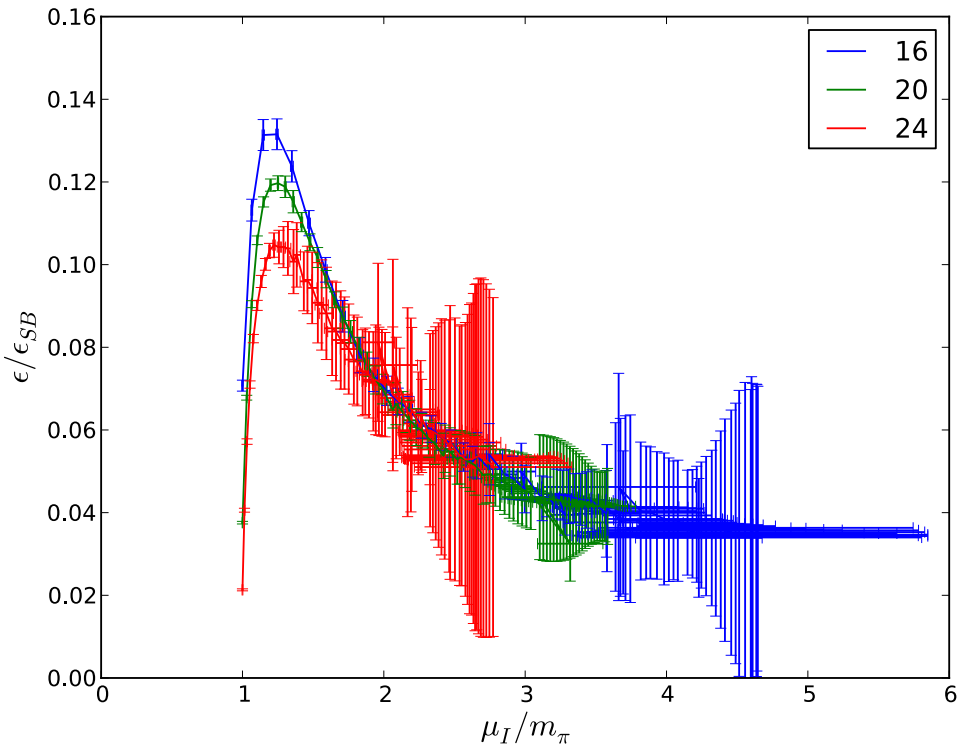


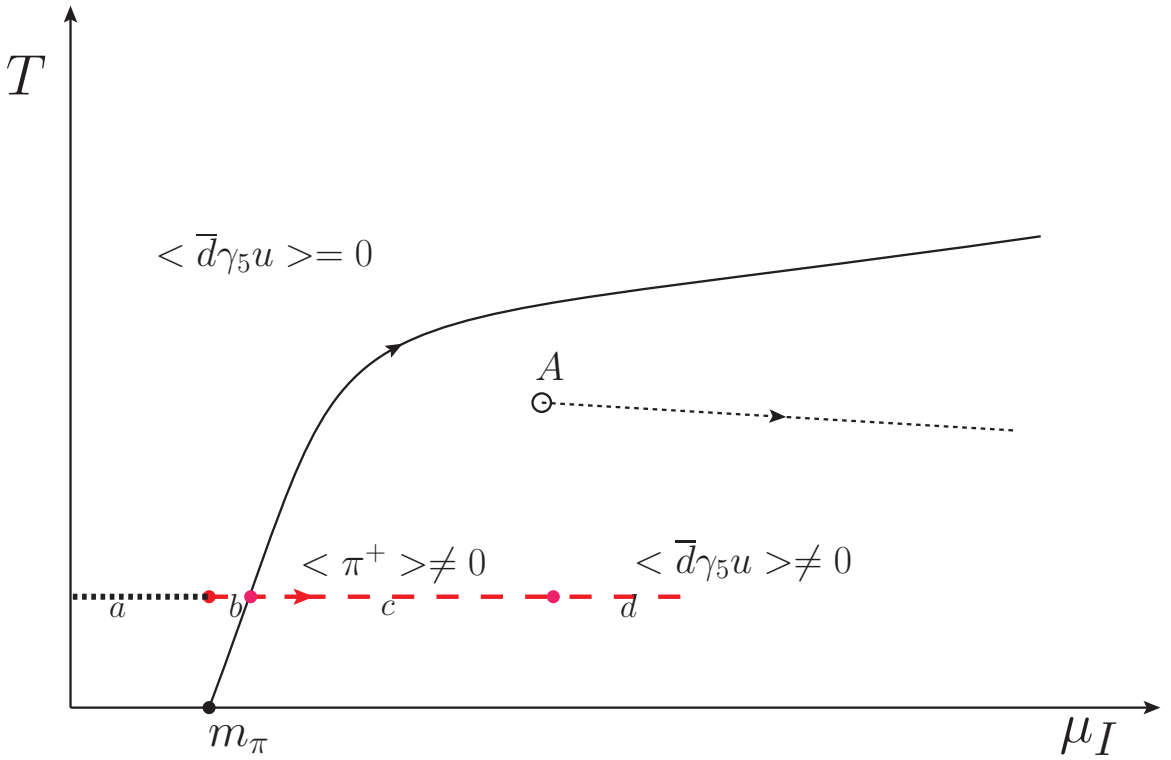
Infinite volume extrapolation



QCD phase diagram

- Compare energy density with the Stefan Boltzmann limit. ($\epsilon_{SB} = 3p_{SB} = \frac{N_f N_c}{4\pi^2}\mu_I^4$ with $N_f = 4, N_c = 3$)





- a Vacuum. ($\mu_I < m_\pi$)
- b Pion gas. ($m_{\pi^+} < \mu_I < 1.3 m_\pi$)
- c BEC. ($1.3 m_\pi < \mu_I < 3 m_\pi$)
- d BCS. ($\mu_I > 3 m_\pi$)**?**

Conclusion and Outlook

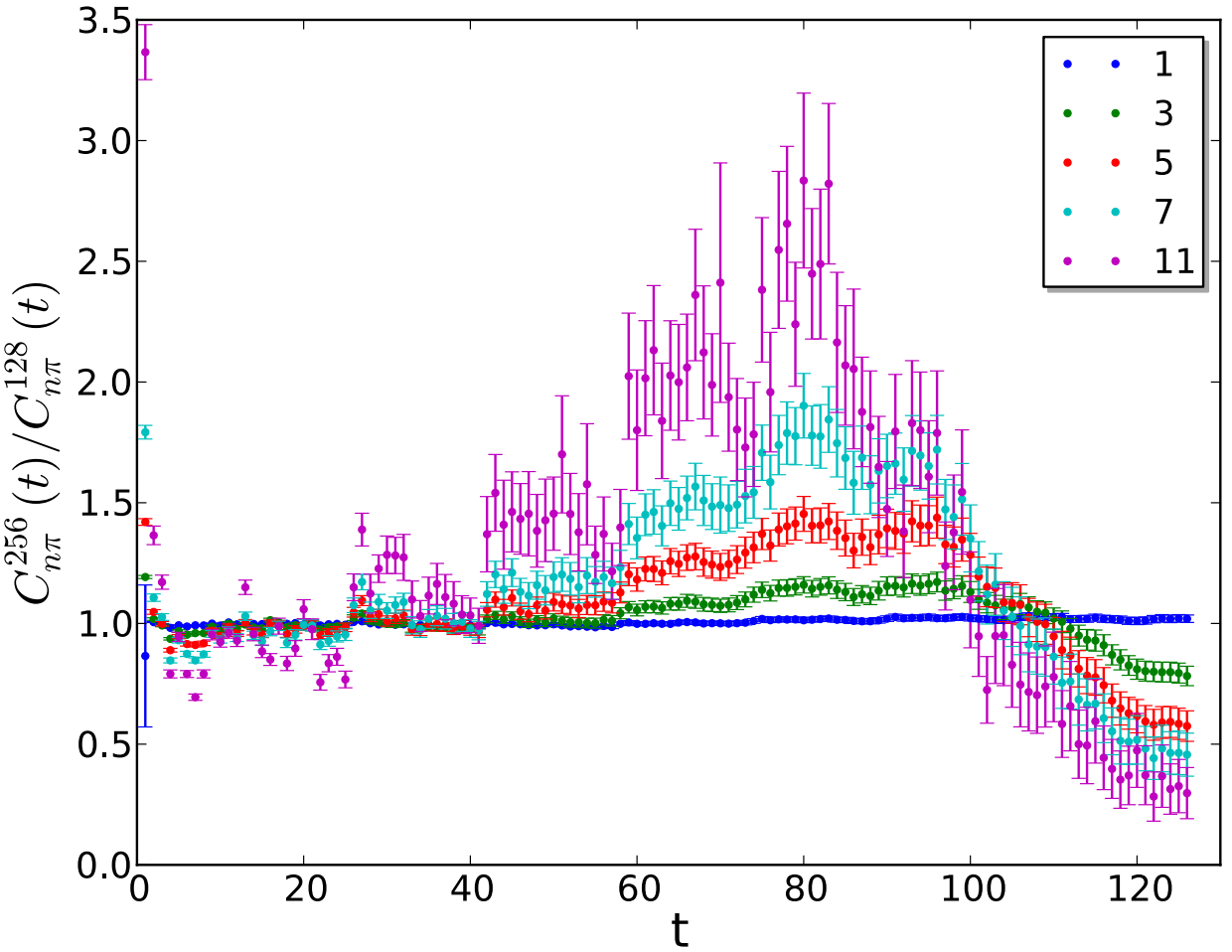
- Construct new algorithms to study many-meson systems.
- Compute correlation functions of systems of quantum number up to 72 π^+ in three different volumes.
- Identify different physical state in the QCD phase diagram.

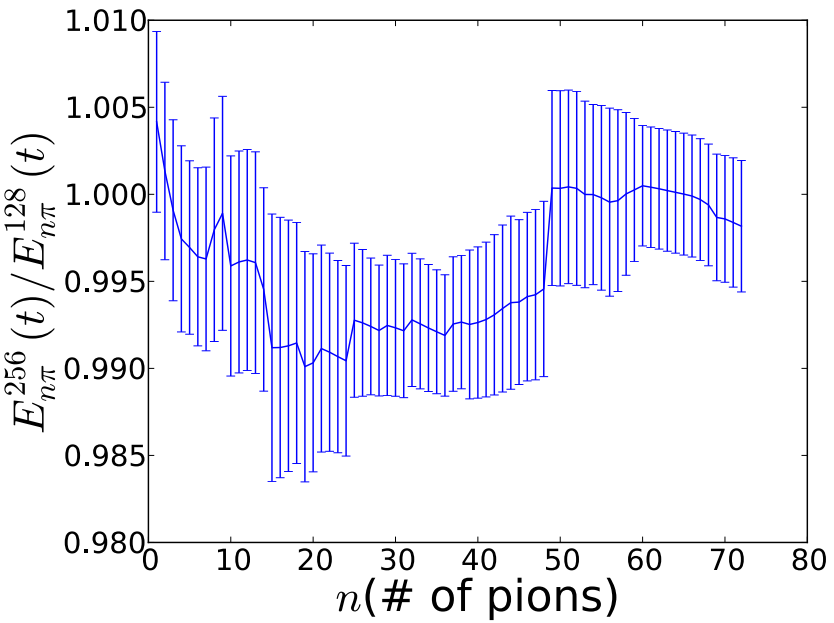
Future: Investigate

- Investigate muti-species systems,
- Suppression of heavy quarkonium spectrum in presence of different isospin density.
- Study other physical quantities around the phase shift.

Thanks for your attention!

Investigate the $A \pm P$ method:





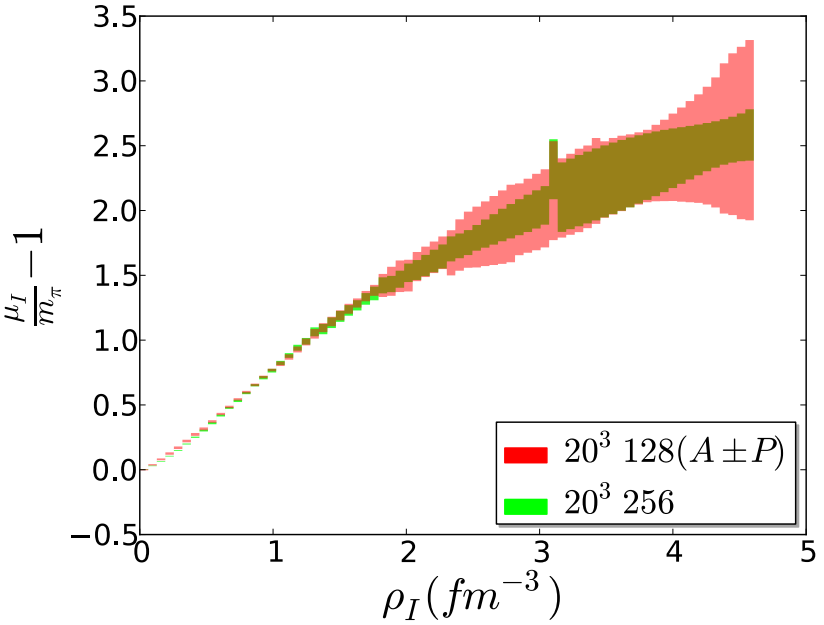


Figure : The ground state energies and isospin chemical potential from ensemble $20^3 \times 128$ with $A \pm P$ method are compared with those from ensemble $20^3 \times 256$

High precision is required. Compare $C_{1\pi^+}$'s calculated from 2 source simulation from Cm and ICm.