Project Report: Mass and Wind

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Abstract

Hurricanes and anti-cyclones can be understood in terms of the unitless quanitity known as the Rossby number. The Rossby number is a ratio of the centrifugal forces to the Coriolis force in a system. Because the Rossby number is unitless, it can be studied across scales of systems so that Rossby numbers for hurricanes are comparable with Rossby numbers for radial inflow laboratory experiments.

1 Introduction

Even though hurricanes are powerful phenomena that warrant close study, they are so large in scale that it is difficult to study them directly. In the laboratory, it is possible to observe water as it drains from a tank in a rotating frame and gather data about the speed at which it spirals relative to the distance from the center. The data can be used to study the effects of the Coriolis force and a centrifugal force and how they balance with a gravitational pressure gradient force. The following analyses will seek to form a supported laboratory model than can be successfully applied to larger scale systems such as hurricanes.

2 Experimental Data: Radial Inflow

The radial inflow experiment is designed to test the the balance of forces aprropriate to a spiralling vortex. The experimental setup includes a bucket with a hole in the center of the bottom plugged with a cork. The bucket is filled with standing water (to remove air bubbles) and is elevated within another enclosure (as seen in Figure 1).

The apparatus is rotated at a constant rate while a camera rotates in synchronization. The apparatus is considered "spun up" when a paper dot can be placed in the bucket and appear stationary from the camera view. This is referred to as solid body rotation. After the apparatus is spun up, the cork is removed so that the water is able to drain away. Paper dots can then be placed in the bucket and tracked using tracking software to determine each particle's position at a given time.

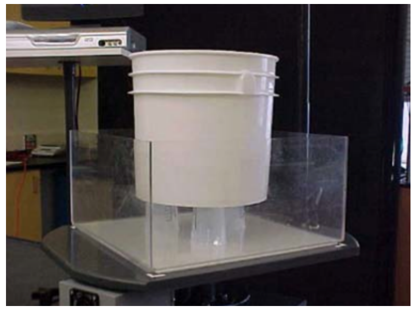

Figure 1: Experimental setup.[1]

2.1 Theory

What can the bucket experiment reveal about the forces present in a rotating frame? The pertinent velocities, for one, can be seen in Figure 2. For an explanation of velocity variables, consider the following:

- The radial velocity is given by v_r.
- The azimuthal velocity is given by v_θ.

Both velocities are relative to the rotating camera and measured in pixels per second. For an explanation of horizontal position variables, consider the following:

- The radius from the center is given by r, in pixels.
- The angle relative to the horizontal is given by θ, in radians.

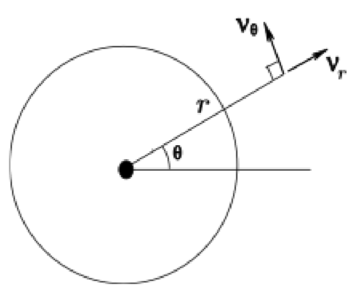

Figure 2: Aerial view of the apparatus.[1]

As the particle is moving azimuthally with velocity v_θ, it is moving with an additional velocity component relative to the observer. Since the bucket is rotating at a constant rate, given by Ω in radians per second, the particle is moving with a component due to rotation. This component is given by Ωr, so that the absolute azimuthal velocity, V_θ is given by

$$V_\theta = v_\theta + \Omega r. \tag{1}$$

Using the conservation of angular momentum it is possible to find a relationship between v_θ and the r. At any point, the angular momentum will be given by $V_\theta r$, which is a constant. Since the relative azimuthal velocity is zero at the edge ($r = R$), the angular momentum is also given by ΩR^2. Setting these equal to one another gives rise to the following relationship:

$$v_\theta = \Omega \left(\frac{R^2 - r^2}{r} \right). \tag{2}$$

As the fluid rotates, its surface takes on a parabolic shape[2] defined by the following:

$$H_0 = \frac{\Omega^2 r^2}{2g}{}^{[1]}, \tag{3}$$

where g is the constant acceleration due to gravity and H_0 is the height of the parabolic surface.[1]

Another governing equation derives from the force of gravity and how it relates to pressure assuming hydrostatic balance. Using this relationship, it is possible to relate the velocities and horizontal position with a vertical position. This relationship is given by

$$p = \rho g(H - z), \tag{4}$$

ρ is the constant density, H is the height of the surface, z is the height (i.e. the bottom is $z = 0$) and p is the difference in pressure from the atmospheric pressure.[1] It is possible to derive a force balance using Equations 1 and 4, obtaining a centrifugal and pressure gradient force, respectively. This balance in the inertial frame is given by

$$\frac{V_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}. \tag{5}$$

When written in terms of the rotating frame from Equation 1 and taking the partial derivative with respect to r of Equation 4, Equation 5 becomes

$$\frac{v_\theta^2}{r} + 2\Omega v_\theta + \Omega^2 r = g\frac{\partial H}{\partial r}. \tag{6}$$

Through manipulation of variables, $\Omega^2 r$ can be rewritten as a partial derivative with respect to r, and the pressure gradient force can be redefined in terms of the surface's divergence from the parabola described by Equation 3. This height difference is given by h and results in a new equation:

$$2\Omega v_\theta + \frac{v_\theta^2}{r} = g\frac{\partial h}{\partial r}. \tag{7}$$

In a more general form, the quantity 2Ω can be replaced by the quantity f, the Coriolis parameter, giving the following:

$$fv_\theta + \frac{v_\theta^2}{r} = g\frac{\partial h}{\partial r}. \tag{8}$$

This equation is known as gradient wind balance, and it is used to describe a system in which the pressure gradient force is balanced by both the Coriolis force, given by $2\Omega v_\theta$, and the outward centrifugal force, given by $\frac{v_\theta^2}{r}$. The the pressure gradient force is balanced largely by the Coriolis force and the outward centrifugal force is negligible, the system is said to be in geostrophic balance; the opposite scenario is known as cyclostrophic balance. A parameter, known as the Rossby number, can be defined to compare the Coriolis and centrifugal forces, and is represented by R_0 in the equation derived from the ratio of the Coriolis force to the centrifugal force[1]:

$$R_0 = \frac{|v_\theta|}{2\Omega r}. \tag{9}$$

When the Rossby number is large ($R_0 >> 1$), the centrifugal force is dominant, but when the Rossby number is small ($R_0 << 1$), the Coriolis force is dominant. When the Rossby number is near one ($R_0 \approx 1$), the forces are in balance and the system is said to be under gradient wind balance. Since it is calculated using experimental parameters, Equation 8 gives the experimental value for the Rossby number.[1] Using Equation 2 from the conservation of angular momentum, a theoretical Rossby number can be calculated from the following:

$$R_0 = \frac{1}{2}\left(\left(\frac{R}{r}\right)^2 - 1\right). \tag{10}$$

2.2 Results

In the experiment, three runs were performed, each with a different Ω (0.5005, 1.000, and 1.500 radians per second). It was difficult to obtain a value for R (the edge of the bucket), so the maximum calculated radius (125 pixels) is used for R. In each case, blue diamonds represent the measured value ofR_0 while red squares represent the theoretical value of R_0 for a given radius.

In the first run, the rotation was set to a constant 1,001 millif, or 0.5005 radians per second. Figure 3 shows Rossby number data for particles under these conditions. Though the high-radius average appears fairly consistent with the theoretical values, the lower-radius averages are not. This difference may indicate a balance of forces slightly different from what was assumed. Since the values are above their theoretical counterparts, it may be that the centrifugal force is playing a larger role than expected.

In the second run, the rotation was set to a constant 2,000 millif, or 1.000 radians per second. Figure 4 shows Rossby number data for particles under these conditions.

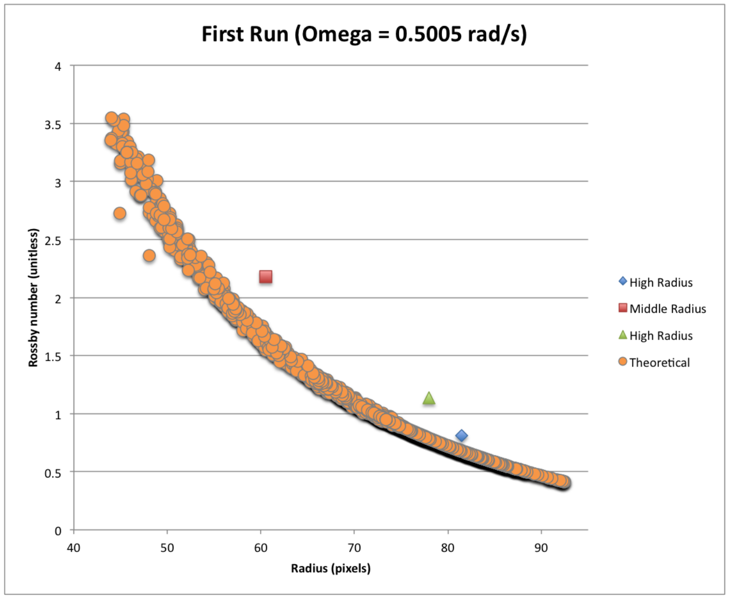

Figure 3: Average measured Rossby numbers and theoretical Rossby number as a function of radius. Notice that the average measured values do not exactly correlate with the theoretical values. This difference may indicate a divergence from the force balance described.

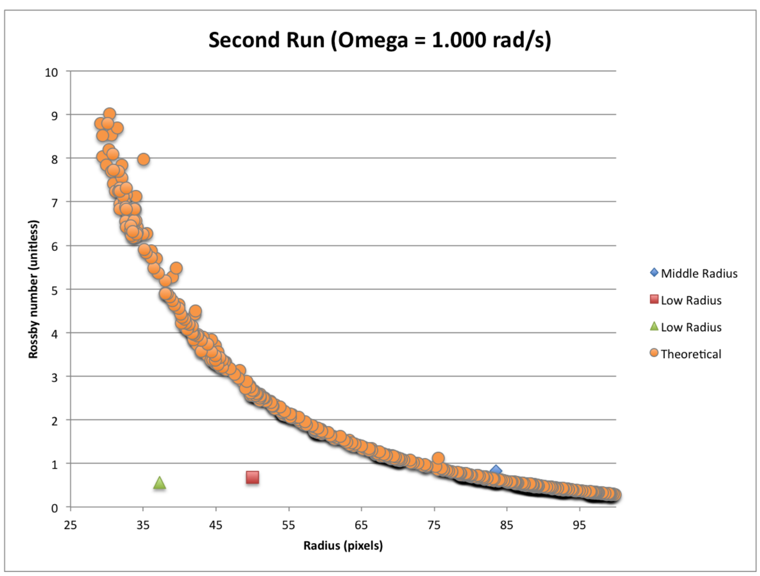

Figure 4: Average measured Rossby numbers and theoretical Rossby number as a function of radius. Notice that the average measured values do not exactly correlate with the theoretical values. This difference may indicate a divergence from the force balance described.

Again, the high-radius average appears fairly consistent with the theoretical values, but the lower-radius averages are not. Since the measured values are lower than their theoretical counterparts, it may be that the Coriolis force is playing a larger role than expected.

In the third and final run, the rotation was set to a constant 3,000 millif, or 1.500 radians per second. Figure 5 shows Rossby number data for particles under these conditions.

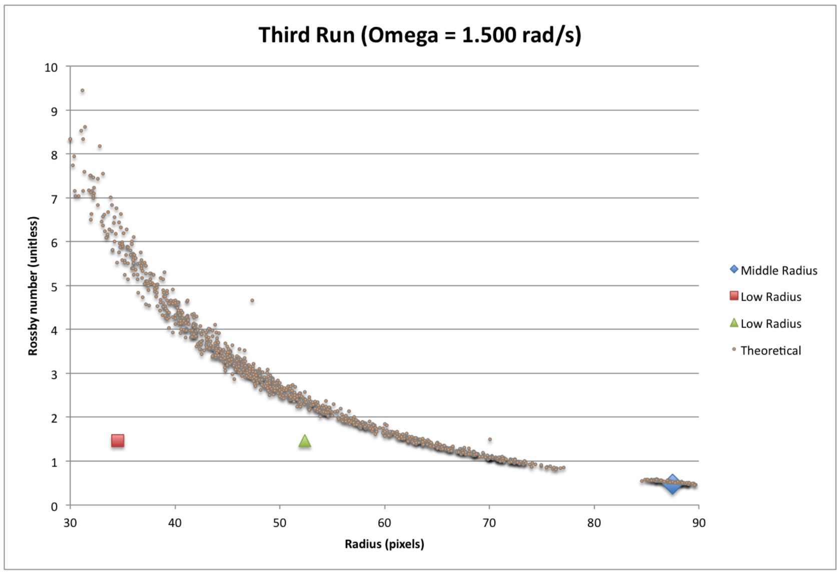

Figure 5: Average measured Rossby numbers and theoretical Rossby number as a function of radius. Notice that the average measured values do not exactly correlate with the theoretical values. This difference may indicate a divergence from the force balance described.

The high-radius average appears to fall along the theoretical curve while the lower-radius values are, again, below it. This, again, may indicate a larger role for the Coriolis force than previously expected.

The measured values roughly, though not exactly, correspond with the theoretical predictions. This is certainly true for the larger-radius averages as noted in Figures 3, 4 and 5. This model does not explain everything exhibited in the experiment, but it does provide a solid framework for predicting the general behavior of this system. In terms of radius, the measurements are accurate within 1.5 pixels. The general trend, though with exceptions, is that the larger radii correspond with lower Rossby numbers and *vice versa*. Since the Rossby number is a non-dimensional quantity, it can be used across systems, including those in the atmosphere.

3 Atmospheric Data

The expectation is that hurricanes should behave similarly to the particles in the laboratory experiment with one modification.

$$f v_\theta + \frac{v_\theta^2}{r} = g\frac{\partial h}{\partial r}. \tag{11}$$

In all governing equations for the lab experiment, Ω is used to represent the rate of rotation. For a hurricane, f will be used such that

$$f = 2\Omega \sin(lat), \tag{12}$$

where lat represents the latitude of the hurricane's center to compensate for the fact that the center of the hurricane is not necessarily aligned with the axis of Earth's rotation. Otherwise, it is expected that the governing equations will be the same. In this case, the Rossby number is given by the following:

$$R_0 = \frac{|v_\theta|}{2r\Omega \sin(lat)}. \tag{13}$$

The hurricane to be considered is Hurricane Florence. The data is from 10 September 2006. Scatterometer data is from 9:50 UTC while GFS analyzed data is from 12:00 UTC. The center of Hurricane Florence is estimated to be 29N, 67W, and the radius is approximated as 550 km. The calculated (lower curve) and theoretical (upper curve) Rossby numbers are shown in Figure 6 based on the Scatterometer data.

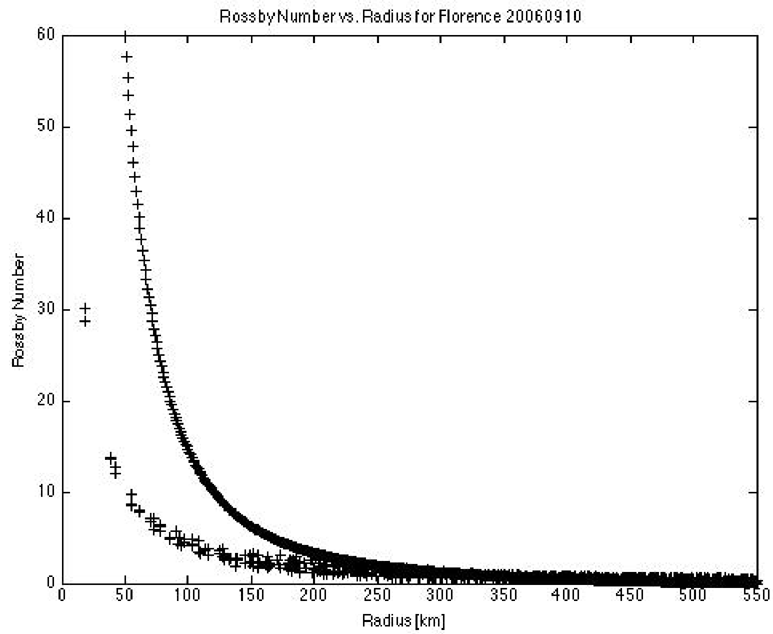

Figure 6: Calculated and Theoretical (Equation 10 with radius of 550 km) Rossby numbers for Hurricane Florence.

Notice that above 250 km, the curves are fairly close to one another. Below 200 km, though, the curves are much more divergent. Based on these curves, Florence is expected to behave geostrophically above 250 km from the center (when the Rossby number is much less than one) and to behave cyclostophically within 50 km (when the Rossby number is much greater than one). For more information, consider the Scalar and Wind data.

Figure 7 displays the observed wind speeds of Hurricane Florence at 850 mb (relatively close to Earth's surface), while Figure 8 displays the geostrophic wind speeds at 850 mb.

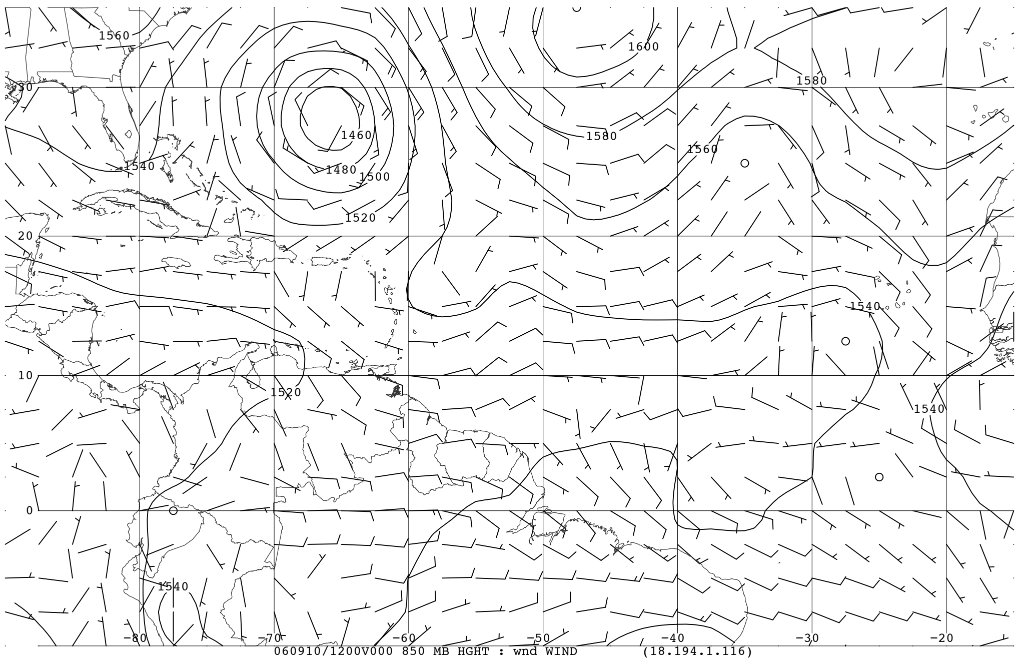

Figure 7: Hurricane Florence, Observed Wind, 850 mb

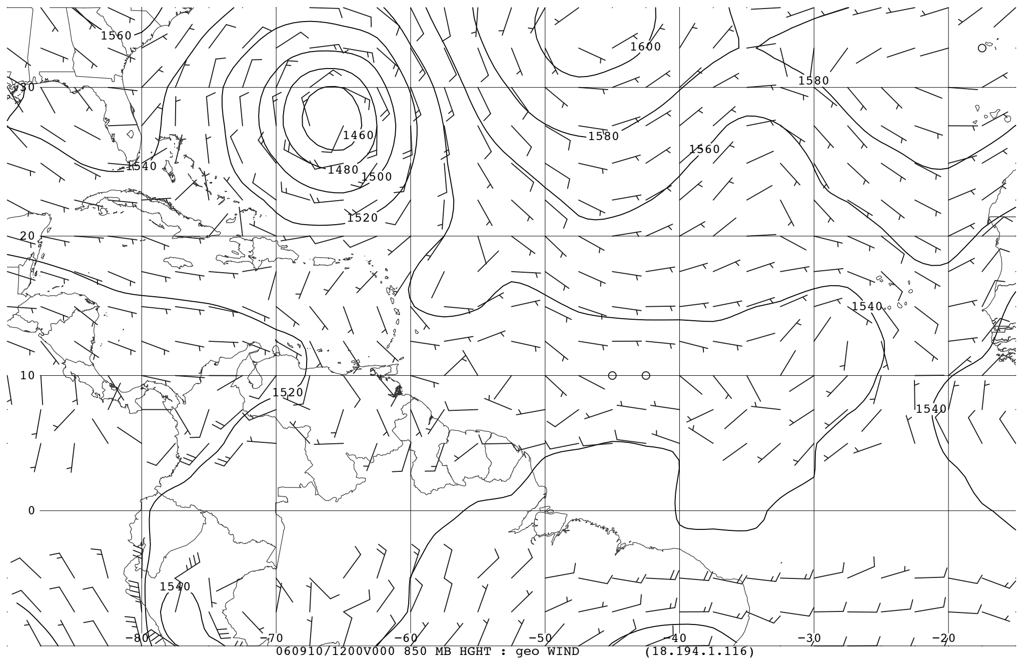

Figure 8: Hurricane Florence, Geostrophic Wind, 850 mb

Wind data for 850 mb is included in Table 1. Notice that for the larger radii, the observed wind is estimated to be the same at the geostrophic wind. This is not the case for the closer radii when the observed wind is better matched by the cyclostrophic wind.

Table 1: Wind speeds at 850 mb for Florence.

r (km)	v_{obv} (m/s)	v_{geo} (m/s)
183	7.5	5
367	10	10
550	7.5	7.5
733	7.5	7.5

Figure 9 displays the observed wind speeds of Hurricane Florence at 500 mb, while Figure 10 displays the geostrophic wind speeds at 500 mb.

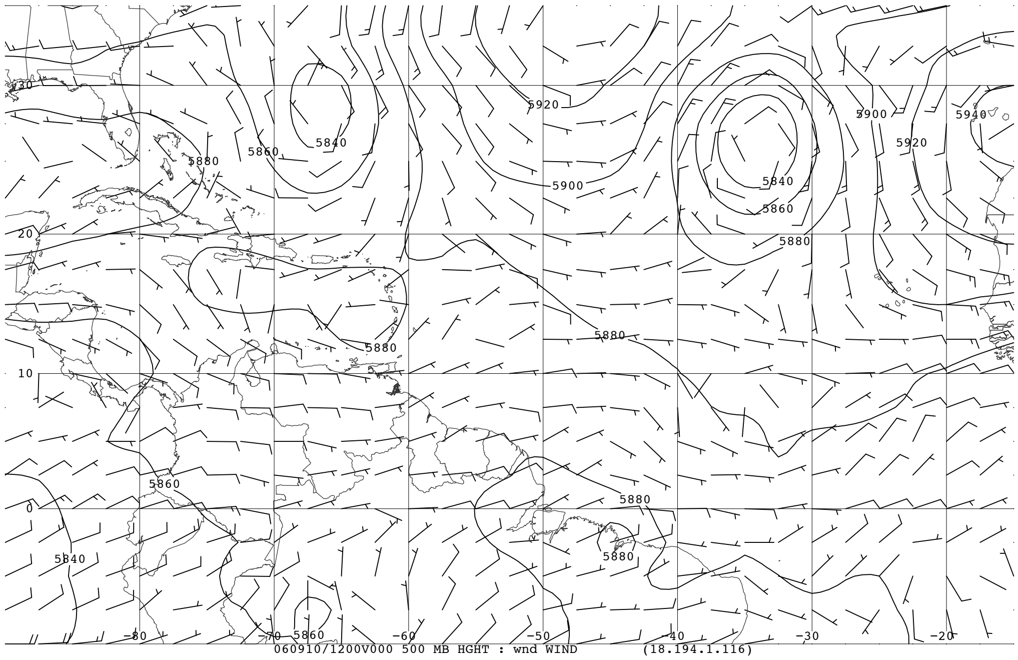

Figure 9: Hurricane Florence, Observed Wind, 500 mb

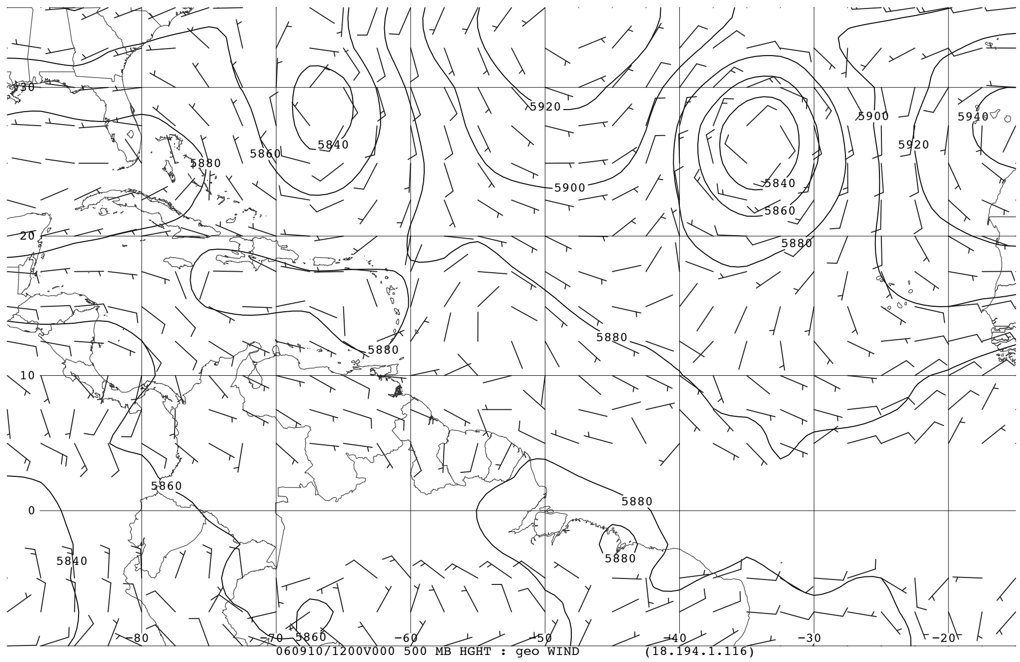

Figure 10: Hurricane Florence, Geostrophic Wind, 500 mb

Notice the second cyclone that has formed. This cyclone, rather than being formed like Florence, is a product of the Jet Stream, and it is stronger at upper levels instead of at the surface (see Figure 11).

Wind data for 500 mb is included in Table 2. In this case, the observed wind matches the geostrophic wind at these larger radii.

Table 2: Wind speeds at 500 mb for Florence.

r (km)	v_{obv} (m/s)	v_{geo} (m/s)
367	5	5
733	5	5

Figure 11 displays the observed wind speeds of Hurricane Florence at 150 mb (far from Earth's surface), while Figure 12 displays the geostrophic wind speeds at 150 mb.

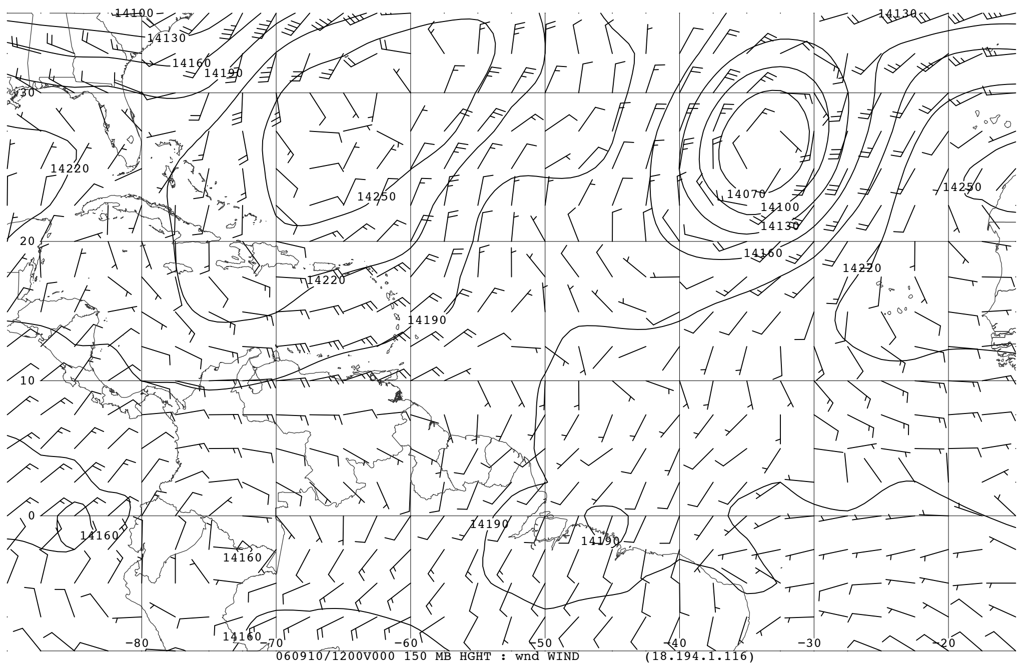

Figure 11: Hurricane Florence, Observed Wind, 150 mb

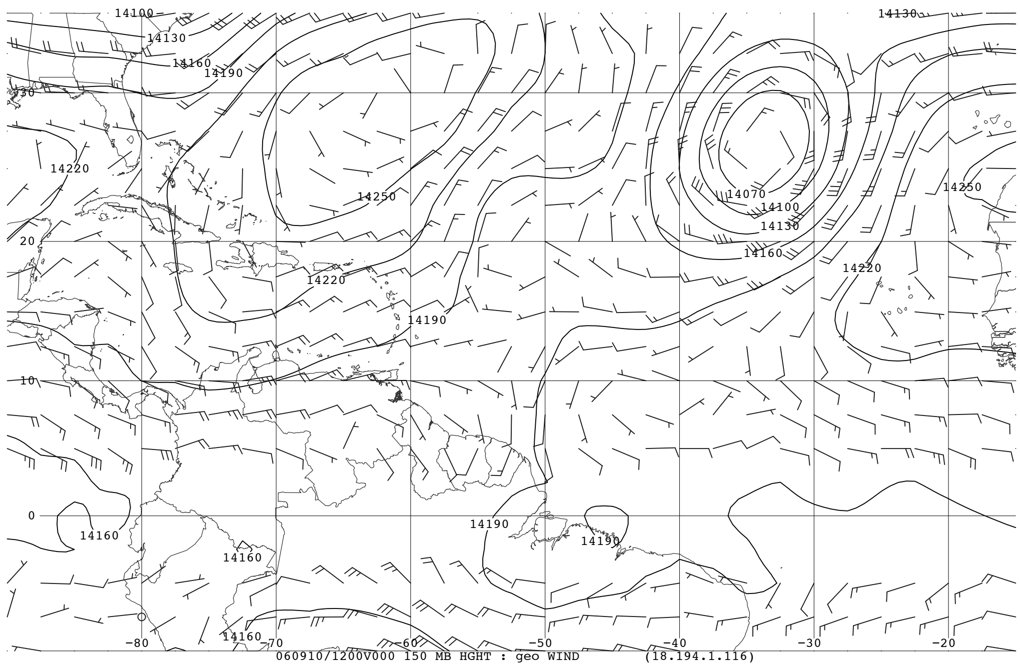

Figure 12: Hurricane Florence, Geostrophic Wind, 150 mb

Notice the gradient for the second cyclone. Wind data for the Jet Stream cyclone is recorded in Table 3.

Table 3: Wind speeds at 150 mb for the Jet Stream cyclone.

r (km)	v_{obv} (m/s)	v_{geo} (m/s)
220	5	10
330	12.5	17.5
440	15	20
550	10	17.5

The differences between the observed and geostrophic wind indicate that this cyclone may not be behaving geostrophically; rather, it may fall under the category of gradient wind.

At 150 mb, a change occurs. At higher pressures, the system behaved cyclonically, spinning in a counterclockwise manner due to the pressure gradient force and opposing Coriolis and centrifugal forces. At this level, the cyclone changes direction in response to a reversal in the pressure gradient, and it becomes an anti-cyclone. This change in direction serves to conserve the angular momentum of the system. Even though the Florence system has changed direction, the Jet Stream cyclone is still rotating cyclonically. This cyclone will instead be weakened in the lower troposphere, and it will sit on top of a surface anti-cyclone.

4 Data Comparison

The laboratory experiment can be treated, essentially, as an upside down hurricane (the fluid vacates down instead of up). The wind data taken from Hurricane Florence at 850 and 500 mb match closely with the experimental data as well as the theory. The Rossby number is a dimensionless quantity that was defined in order to describe these systems as a function of the distance from the center of the system. Both systems are studied in terms of the Rossby number.

That is to say, at large radii, both systems exhibited low Rossby numbers. Because of the way the Rossby number is defined, these systems were exhibiting a stronger Coriolis force than centrifugal force. As a result, both were in geostrophic balance at the higher radii. Though low radii data were difficult to obtain for Hurricane Florence, the intermediate piece of data obtained fits in with the model, though, again, it is only one data point. Interestingly, the hurricane data fit the model better than the laboratory data in terms of the Rossby number.

One distinction to be made is the anti-cyclonic behavior exhibited in the atmospheric system. This phenomena was not observed in the laboratory experiment, but that is due to the way the experiment was designed. The laboratory setup was not designed to exhibit anti-cyclonic behavior, so it did not; however, it may be to create such an experiment to display this high pressure system pattern by utilizing a lid to stop the cyclonic motion. Another important distinction is the influence of friction. In the laboratory experiment, friction does not play a large role in the behavior of the system until the water level is sufficiently low; for the larger atmospheric system friction plays a much larger role, and a model to accurately depict the system would need to incorporate friction in some manner.

5 Conclusion

The radial inflow experiment provides an accessible and accurate way to observe and analyze the behavior of large scale systems such as hurricanes. Even though these two systems are separated by several orders of magnitude, they follow nearly identical governing equations. These governing equations derive from the balance of forces within the radial inflow laboratory experiment and withstand the tests of observation and analysis. These equations are tested on large atmospheric systems to evaluate their worth and the model's worth.

Through initial analyses, the model persists. Both cases, laboratory and atmospheric, demonstrate that the pressure gradient force is balanced at high radii by the Coriolis force, indicated by the low Rossby number. Though not definitive, the data also suggest that the pressure gradient force is balanced at low radii by a centrifugal force, indicated by a high Rossby number.

6 Suggestions For Improvement

Though this model has been shown accurate, to a degree, with more experimentation and more precise analyses, it is possible to improve upon this model and more accurately describe both the atmosphere and the laboratory experiment.

One of the first problems noticed was that the particle tracker was tracking too many particles at once. In later runs, this problem was alleviated by using one or two particles at a time. Another problem with the particle tracker came with long tracks. Whenever a single particle was following a path for a certain length of time, the tracker would "reset" and consider the remainder of the track as belonging to a new particle. This made it difficult to record and analyze full paths from start to finish. Improving the particle tracker to prevent this problem would provide more and better data. A higher resolution particle tracker would also allow for more precise and accurate data.

In performing the analyses on both the laboratory and atmospheric data, one of the more difficult (and error-bearing) problems was estimating centers. For the particle tracker, it was much simpler since it considered the hole in the bottom (i.e. the center) to be a particle and tracked it as such. Unfortunately, this is another problem in itself. In regards to the hurricane data, the center was much more difficult to identify with much accuracy. Future analyses that can provide better determination of this feature would improve the experiment.

In both the experiment and the atmospheric data, there were cases where the calculated Rossby number was significantly lower than the theoretical value and where the calculated value was considerably higher, which may indicate some factors that had not been considered. When collecting data and designing experiments for this problem in the future, it may be useful to consider how friction would affect the smaller system, even if it is negligible. This would make it easier, or perhaps just possible, to consider the importance of friction in the behavior of hurricanes.

References

[1] Marshall, John. "12.307 Project 1: Radial Inflow Experiment."

[2] Marshall, John and Alan Plumb. "Chapter 6: The equations of fluid motion." Available at http://paoc.mit.edu/labweb/notes/chap6.pdf.