Spring 2008 - 6.002
Pset #8 Solutions

Exercise 8.1

$i_{IN}(t) = 1\text{ mA } u(t)$

$v_{IN}(t) = 1\text{V}\left(1 + e^{-t/\mu s}\right) u(t)$

$v_{IN}(0) = 2\text{V}$ @ $t = 0^+$ all the current is going through the resistor, R_1

$$\boxed{R_1 = \frac{2\text{ V}}{1\text{mA}} = 2\text{ k}\Omega}$$

$v_{IN}(\infty) = 1\text{ V}$ @ $t \to \infty$ the inductor is an short and R_1 & R_2 are in parallel

$$\frac{R_1 R_2}{R_1 + R_2} = \frac{1\text{V}}{1\text{mA}} = 1\text{ k}\Omega$$

$$\boxed{R_2 = 2\text{ k}\Omega}$$

$$\tau = 1\mu s = \frac{L}{R_1 + R_2} \Rightarrow L = (1\mu s)(4\text{ k}\Omega)$$

$$\boxed{L = 4\text{ mH}}$$

Exercise 8.2

@ $t = 0^+$ the capacitor looks like a short.

$$v_{IN}(0^+) = \left(\tfrac{1}{2}\text{ k}\Omega\right)(1\text{mA}) = \tfrac{1}{2}\text{ V}$$

@ $t \to \infty$ the capacitor is an open

$$v_{IN}(\infty) = (1\text{k}\Omega)(1\text{mA}) = 1\text{V}$$

$$\tau = R_{tot} C = (2\text{ k}\Omega)\left(\tfrac{1}{2}\mu F\right) = 1 \times 10^{-3}\text{ s}$$

$$\boxed{v_{IN}(t) = \left[\tfrac{1}{2}\text{V} + \tfrac{1}{2}\text{V}\left(1 - e^{-t/1.0 \times 10^{-3}\text{ s}}\right)\right] u(t)}$$

Problem 8.1:

(A) $v(0^+) = v(0^-) = 0$

$i(0^+) = v(0^-) = 0$ No infinite current/voltage

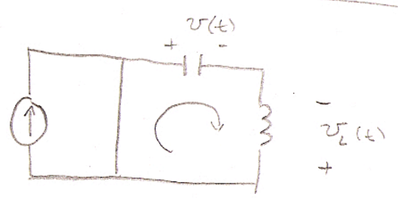
$$i_c(t) = I = C\frac{dv(t)}{dt} \Longrightarrow v(t) = \frac{1}{C}\int_{0^+}^{t} i_c(\tau)\,d\tau + \cancel{v(0^+)}^{\to 0}$$

$$= \frac{It}{C}$$

$$v_L(t) = 0 \Rightarrow \underbrace{i(t) = k}_{\text{constant}} \Rightarrow i(t) = 0$$

$$\boxed{\begin{array}{ll} v(t) = \dfrac{I \cdot t}{C} & \\ i(t) = 0 & \text{for } 0 \le t \le T_1 \end{array}}$$

(B)



KVL: $v_L(t) = v(t) = L\frac{di}{dt}$

$$v(t) = L\frac{d}{dt}\left(-C\frac{dv(t)}{dt}\right)$$

$$LC\frac{d^2v(t)}{dt^2} + v(t) = 0$$

$$\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$$

Let $\omega_0 = \frac{1}{\sqrt{LC}}$

Choose: $v(t) = A\cos(\omega_0 t')$ $t' = t - T_1$

$v(T_1) = A = \frac{IT_1}{C}$ Plugging into part (a)

$$\therefore \quad \boxed{v(t) = \frac{I T_1}{C} \cos[\omega_o (t - T_1)] \quad , \quad T_1 \le t \le T_2}$$

$$i(t) = -C \frac{dv(t)}{dt} = -C\left(-\frac{I T_1 \omega_o}{C}\right) \sin(\omega_o (t - T_1))$$

$$\boxed{i(t) = I T_1 \omega_o \sin[\omega_o (t - T_1)] \quad , \quad T_1 \le t \le T_2}$$

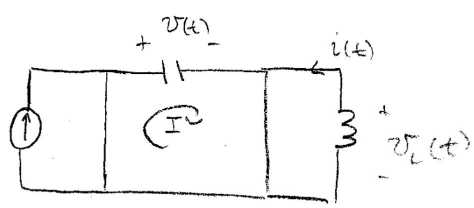
when $v(t) = 0$, $t = T_2$

$$\sin(\omega_o (T_2 - T_1)) = 0$$

$$\therefore \quad \omega_o [T_2 - T_1] = \frac{\pi}{2}$$

$$\boxed{T_2 = T_1 + \frac{\pi}{2\omega_o}}$$

(c)



$v_L(t) = 0 \Rightarrow i(t) = \text{constant}$

$$\boxed{i(t) = i(T_2) = I T_1 \omega_o \sin[\omega_o (T_2 - T_1)] \quad \text{for} \quad T_2 \le t \le T_3}$$

doing KVL around the loop I tells us that $v(t)$ must equal 0.

$$\boxed{v(t) = 0 \quad , \quad T_2 \le t \le T_3}$$

(D) @ $t = T_3$, $v(T_3) = 0$

$$\Rightarrow \boxed{v(t) = \frac{I}{C}(t - T_3) , \quad T_3 \le t \le T_4}$$

$v_L(t) = 0$, $i(t)$ must remain constant

$$\boxed{i(t) = I\, T_1\, \omega_0 \qquad T_3 \le t \le T_4}$$

(E) @ $t = T_4$, $v(T_4) = \frac{I}{C}(T_4 - T_3)$

$i(T_4) = I\, T_1\, \omega_0$

Similar to Part B:

$$\frac{d^2 v(t)}{dt^2} + \frac{1}{LC} v(t) = 0 \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

The general solution see in class is:

$$v(t) = A_1 \cos[\omega_0(t - T_4)] + A_2 \sin[\omega_0(t - T_4)]$$

$$v(T_4) = \frac{I}{C}(T_4 - T_3) \Rightarrow A_1 = \frac{I}{C}(T_4 - T_3)$$

$$i(t) = -C\frac{dv(t)}{dt} \Rightarrow i(T_4) = -C\frac{dv(t)}{dt} = I\, T_1\, \omega_0$$

$$\frac{dv(t)}{dt} = -\frac{I\, T_1\, \omega_0}{C}$$

$$\frac{dv(t)}{dt} = -A_1 \omega_o \sin[\omega_o (t - T_4)] + A_2 \omega_o \cos[\omega_o (t - T_4)]$$

$$\frac{dv(T_4)}{dt} = +A_2 \omega_o = -\frac{I T_1}{C} \omega_o$$

$$A = -\frac{I T_1}{C}$$

$$\boxed{\begin{aligned} v(t) &= \frac{I}{C}(T_4 - T_3)\cos[\omega_o (t - T_4)] - \frac{I T_1}{C}\sin[\omega_o (t - T_4)] \\ i(t) &= I \omega_o \left[(T_4 - T_3)\sin[\omega_o (t - T_4)] + T_1 \cos[\omega_o (t - T_4)] \right] \\ &\text{for} \quad T_4 \le t \le T_5 \end{aligned}}$$

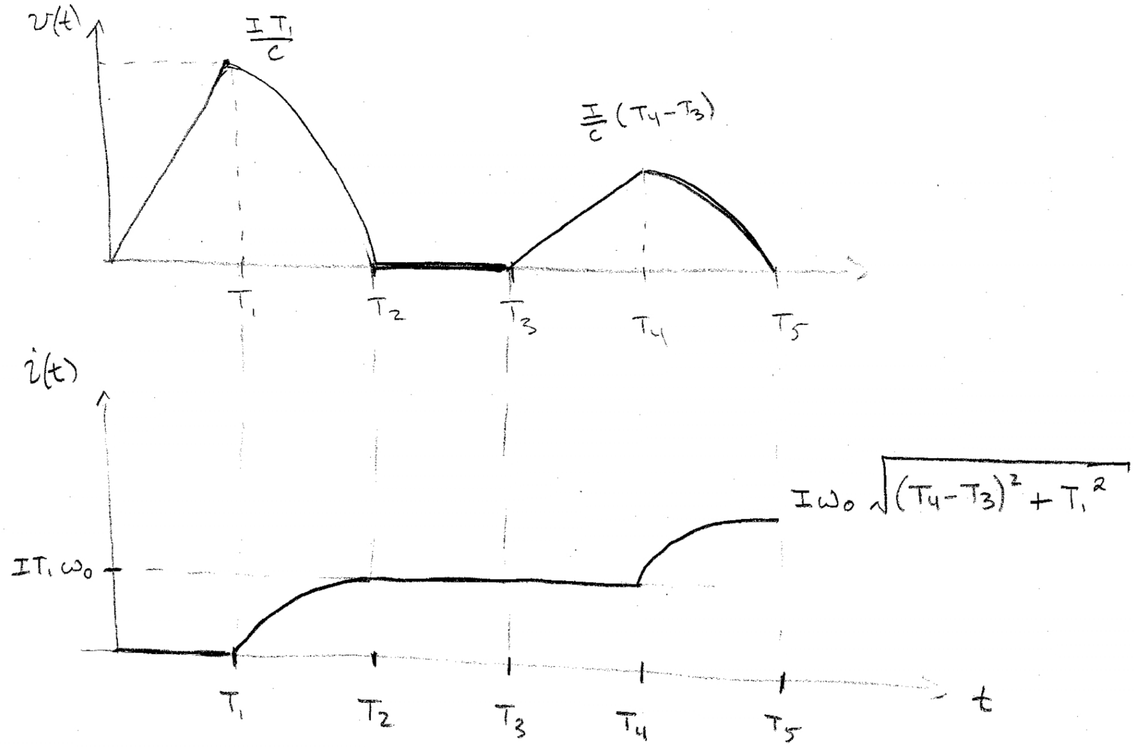
$$v(T_5) = 0$$

$$\Rightarrow \frac{I}{C}(T_4 - T_3)\cos[\omega_o (T_5 - T_4)] = \frac{I T_1}{C}\sin[\omega_o (T_5 - T_4)]$$

$$\tan(\omega_o (T_5 - T_4)) = \frac{T_4 - T_3}{T_1}$$

$$\boxed{T_5 = T_4 + \frac{1}{\omega_o}\tan^{-1}\left(\frac{T_4 - T_3}{T_1}\right)}$$

(F)


v(t)
$\frac{IT_1}{C}$
$\frac{I}{C}(T_4-T_3)$
T_1 T_2 T_3 T_4 T_5
i(t)
$I\omega_0\sqrt{(T_4-T_3)^2+T_1^2}$
$IT_1\omega_0$
T_1 T_2 T_3 T_4 T_5
t

em 8.2

$$E_c = \frac{1}{2} C v^2(T_1) = \frac{1}{2} C \frac{I^2 T_1^2}{C^2} = \frac{1}{2} \frac{I^2 T_1^2}{C}$$

$$\boxed{E_c = \frac{1}{2} \frac{I^2 T_1^2}{C}}$$

$$E_L = \frac{1}{2} L i^2(T_2) = \frac{1}{2} \frac{I^2 T_1^2}{C}$$

$$i^2(T_2) = \frac{I^2 T_1^2}{LC} = \frac{I^2 T_1^2}{1/\omega_0^2}$$

$$i^2(T_2) = I^2 T_1^2 \omega_0^2$$

$$i(T_2) = \pm I T_1 \omega_0$$

Since the inductor is being charged w/ positive $v_c(t)$, we choose the positive solution

$$\boxed{i(T_2) = I T_1 \omega_0}$$

(c) @ $t = T_4$, $v(T_4) = \frac{I}{C}(T_4 - T_3)$

$$E_c = \frac{1}{2} C V^2(T_4) = \frac{1}{2} C \frac{I^2 (T_4 - T_3)^2}{C^2}$$

$$\boxed{E_c = \frac{1}{2} \frac{I^2 (I_4 - I_3)^2}{C}}$$

(D) at T_5 the energy from the capacitor is transfered to the inductor which already had $\frac{1}{2}\frac{I^2 T_1^2}{C}$ stored.

$$\therefore \quad E_L(T_5) = \frac{1}{2} L i^2(T_5)$$

$$\frac{1}{2} L i^2(T_5) = \frac{1}{2}\frac{I^2 T_1^2}{C} + \frac{1}{2}\frac{I^2 (T_4 - T_3)^2}{C}$$

↑
Part (c)

$$\therefore \quad \boxed{i_L(T_5) = I \omega_o \sqrt{(T_4 - T_3)^2 + T_1^2}}$$

(E) For each cycle, $v_c = \frac{I}{C} T$ after S_1 is open

At this point $E_c = \frac{1}{2} C \left(\frac{I}{C} T\right)^2 = \frac{1}{2}\frac{I^2 T^2}{C}$

In the 2^{nd} part of the cycle, this is transferred to the inductor:

After 1^{st} cycle: $E_L = \frac{1}{2}\frac{I^2 T^2}{C}$

2^{nd} cycle $E_L = 2 \cdot \frac{1}{2}\frac{I^2 T^2}{C}$

Afte n Cycles $E_L = \frac{n}{2}\frac{I^2 T^2}{C} = \frac{1}{2} L i_L^2(nT)$

$$i_L^2(nT) = \frac{n I^2 T^2}{LC}$$

$$\Rightarrow \boxed{i_L(nT) = I T \omega_o \sqrt{n}}$$

Problem 8.3

(A) See attached graph

(B) $\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{1.32ms - 1.2ms}$

$$\boxed{\omega = 52.4 \times 10^3 \ \frac{rad}{s}}$$

$$\boxed{\tau \simeq 0.18 \ ms}$$

(C) Using KVL:

$$v_{IN} = v_R + v_L + v_C$$

$$= i \cdot R + L\frac{\partial i}{\partial t} + v_c \qquad i = C\frac{\partial v_c}{\partial t}$$

$$\frac{\partial v_{IN}}{\partial t} = R\frac{\partial i}{\partial t} + L\frac{\partial^2 i}{\partial t^2} + \frac{i}{c}$$

$$\boxed{\frac{\partial^2 i}{\partial t^2} + \frac{R}{L}\frac{\partial i}{\partial t} + \frac{i}{LC} = \frac{1}{L}\frac{\partial v_{IN}}{\partial t}}$$

(D) If the network is initially at rest. $i = 0$

$$\boxed{\begin{array}{lll} i(0^-) = 0 & v_c(0^-) = V_{IN1} & v_L(0^-) = 0 \\ v_R(0^-) = 0 & \frac{di}{dt}(0^-) = 0 & \end{array}}$$

Just after the step, i will still be zero

$$\boxed{\begin{array}{lll} i(0^+) = 0 & v_c(0^+) = V_{IN1} & v_L(0^+) = V_{IN2} - V_{IN1} \\ v_R(0^+) = 0 & \frac{di}{dt}(0^+) = \frac{V_{IN2} - V_{IN1}}{L} & \end{array}}$$

(E) for $t \to \infty$ and a constant input, the capacitor looks like an open circuit. $\Rightarrow i \to 0$

$\therefore \; i(\infty) = 0$

As we saw in Lecture an RLC circuit will ring like sin( ) but amplitude of i will decay. This gives the form.

$$i(t) = I \sin(\omega t + \phi) e^{-\alpha t}$$

for $i(0) = I \sin(\phi) = 0 \quad \therefore \; \boxed{\phi = 0}$

$$\boxed{\begin{aligned} \alpha &= \frac{R}{2L} \\ \omega = \omega_d &= \sqrt{\omega_0^2 - \alpha^2} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \end{aligned}}$$

$$\left.\frac{di}{dt}\right|_{0^+} = I\omega \cos(0) = \frac{v_L(0^+)}{L} = \frac{V_{IN2}}{L} = \frac{V_{IN2} - V_{IN1}}{L}$$

$$\boxed{I = \frac{V_{IN2} - V_{IN1}}{\omega L}}$$

(F)

$$v_{IN} = v_R + v_{out}$$

$$v_{IN} = i(t) \cdot R + v_{out}$$

For $V_{IN1} \to V_{IN2}$: $\quad v_{out} = V_{IN2} - IR\sin(\omega t)e^{-\alpha t}$

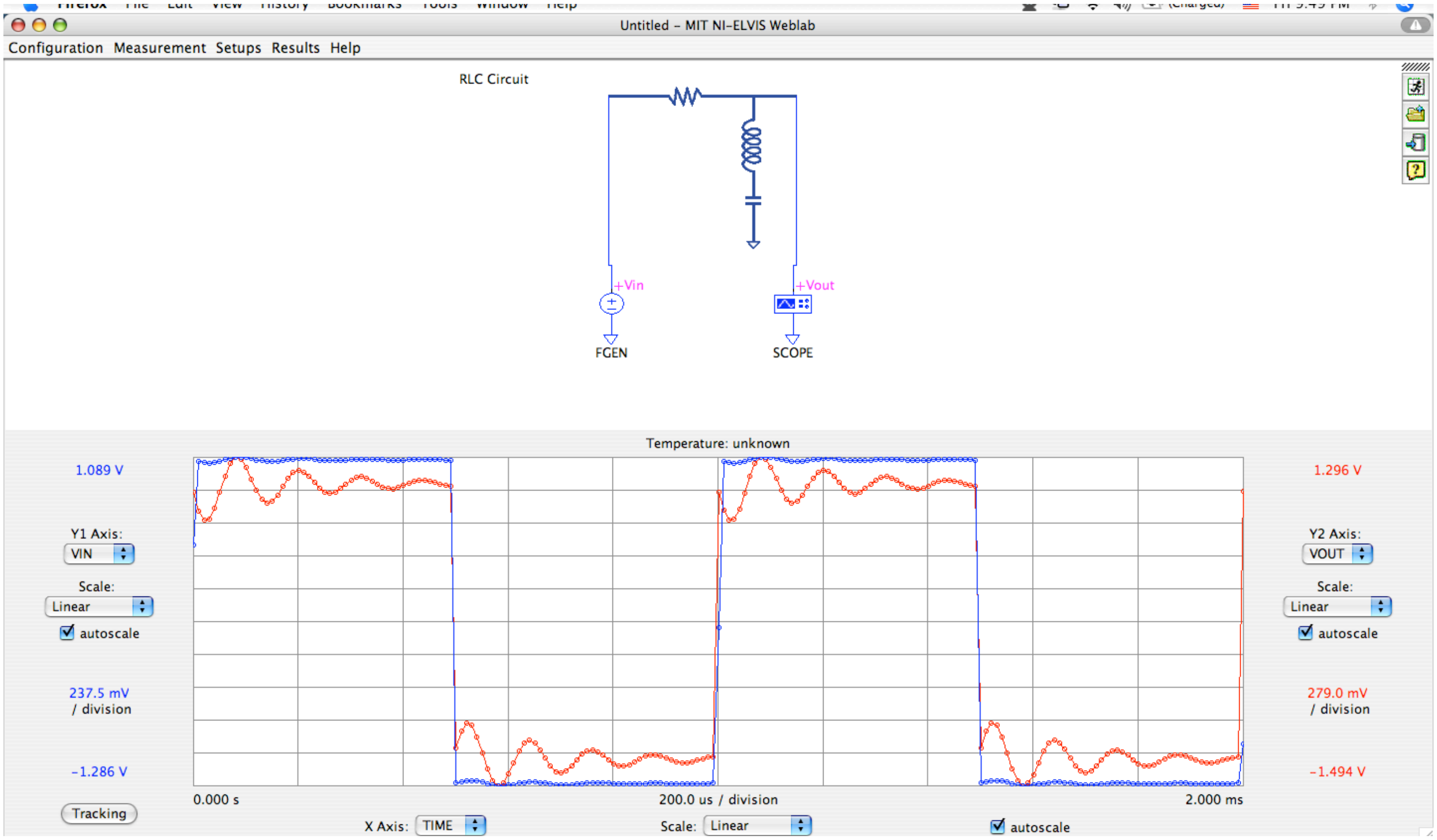
For $V_{IN2} \to V_{IN1}$ $\quad v_{out} = V_{IN1} + IR\sin(\omega t)e^{-\alpha t}$

(11)

(G) $\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 52.4 \times 10^3$

$\alpha = \frac{R}{2L} = 0.18 \times 10^{-3}$

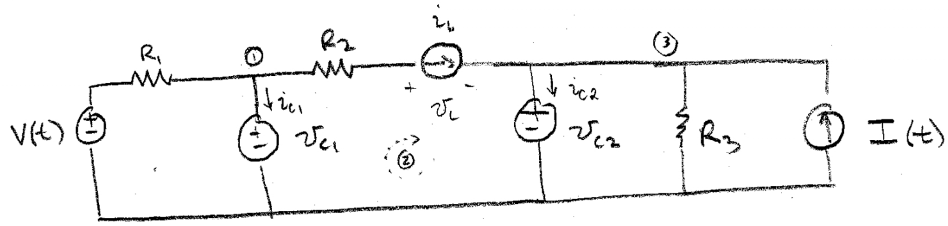
$\frac{1}{LC} = 2.75 \times 10^9$


Untitled – MIT NI-ELVIS Weblab
Configuration Measurement Setups Results Help
RLC Circuit
+Vin
+Vout
FGEN
SCOPE
Temperature: unknown
1.089 V
Y1 Axis:
VIN
Scale:
Linear
autoscale
237.5 mV
/ division
-1.286 V
Tracking
1.296 V
Y2 Axis:
VOUT
Scale:
Linear
autoscale
279.0 mV
/ division
-1.494 V
0.000 s
200.0 us / division
2.000 ms
X Axis: TIME
Scale: Linear
autoscale

Problem 8.3A

Problem 8.4

(A)



(B)

@ 1: $\frac{V(t) - v_{c1}}{R_1} - i_{c1} - i_L = 0$

$$\boxed{i_{c1} = \frac{V(t) - v_{c1}}{R_1} - i_L}$$

@ 2: $v_{c1} - i_L R_2 - v_L - v_{c2} = 0$

$$\boxed{v_L = v_{c1} - i_L R_2 - v_{c2}}$$

@ 3: $i_L - i_{c2} - \frac{v_{c2}}{R_3} + I(t) = 0$

$$\boxed{i_{c2} = i_L + I(t) - \frac{v_{c2}}{R_3}}$$

(C) $v_L = L \frac{di_L}{dt}$

$i_c = C \frac{dv_c}{dt}$

From above:

$$C \frac{dv_{c1}}{dt} = \frac{V(t) - v_{c1}}{R_1} - i_L$$

$$L \frac{di_L}{dt} = v_{c1} - i_L R_2 - v_{c2}$$

$$C \frac{dv_{c2}}{dt} = i_L + I(t) - \frac{v_{c2}}{R_3}$$

$$\frac{d}{dt}\begin{bmatrix} i_c(t) \\ v_{c1}(t) \\ v_{c2}(t) \end{bmatrix} = \begin{bmatrix} -R_2/L & 1/L & -1/L \\ -1/C_1 & -\frac{1}{R_1C_1} & 0 \\ 1/C_2 & 0 & -\frac{1}{C_2R_3} \end{bmatrix}\begin{bmatrix} i_c(t) \\ v_{c1}(t) \\ v_{c2}(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{R_1C_1} \\ 0 & \frac{1}{C_2} \end{bmatrix}\begin{bmatrix} I(t) \\ V(t) \end{bmatrix}$$