# Problem Set 10 Solutions

**E10.1** The Key idea is to realize that both $v_1$ and $v_2$ can be obtained through voltage dividers:

$$\hat{V}_1 - \hat{V}_2 = V\left[\frac{R}{R+j\omega L} - \frac{1/j\omega C}{R + 1/j\omega C}\right]$$

$$= V\left[\frac{R}{R+j\omega L} - \frac{1}{1+j\omega RC}\right]$$

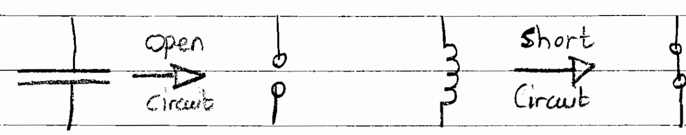
$$= V\,\frac{R + j\omega R^2C - R - j\omega L}{(R+j\omega L)(1+j\omega RC)}$$

$$\hat{V}_1 - \hat{V}_2 = 0 \Rightarrow j\omega R^2C - j\omega L = 0$$

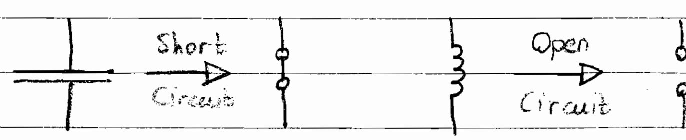
$$j\omega R^2C = j\omega L$$

$$\boxed{L = R^2C}$$

**E10.2** At low frequency:




At high frequency:




(a) LPF: For low freq, $v_O = v_I$. For high freq. $v_O \approx 0V$.

(b) HPF: For low freq, $v_O \approx 0V$. For high freq. $v_O$ is divided down version of $v_I$.

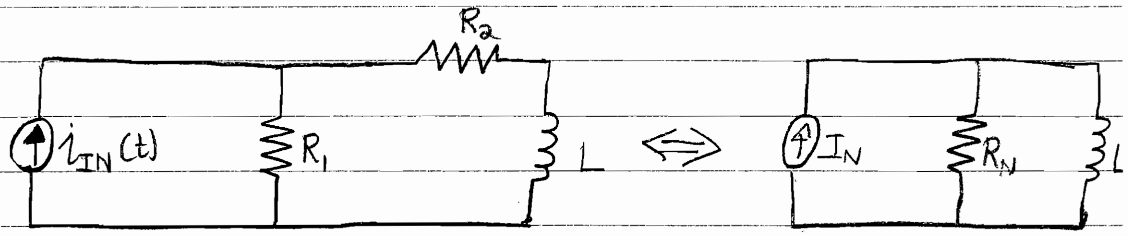
(c) BSF: The LC network is open circuit at low and high freq. At $\omega = \frac{1}{\sqrt{LC}}$, $Z = 0$ which is a short. Hence close to $\omega$, $v_O$ is attenuated.

(d) BSF: The LC network is short circuit at low and high freq. at $\omega = \frac{1}{\sqrt{LC}}$, $Z = \infty$ which is an open. Hence, $v_O$ is again attenuated close to $\omega$.

# Problem Set 10 Solutions

P10.1 (A) $V_{OUT}(t) = i_L(t) R_2$

The current through L is the same as that through $R_2$. Therefore, to find $i_L$ re-draw the circuit as follows:




$$I_N = \frac{R_1}{R_1+R_2} i_{IN}(t) \qquad R_N = R_1 + R_2$$

$$I_N = \frac{v_L(t)}{R_N} + i_L(t)$$

$$I_N = \frac{L}{R_N} \frac{d i_L(t)}{dt} + i_L(t)$$

$$I_N R_2 = \frac{L}{R_N} \cdot \frac{d V_{OUT}(t)}{dt} + V_{OUT}(t)$$

$$\frac{R_1 R_2}{R_1+R_2} i_{IN}(t) = \frac{L}{R_1+R_2} \frac{d V_{OUT}(t)}{dt} + V_{OUT}(t)$$

(B)

$$\frac{R_1 R_2}{R_1+R_2} I_{in} e^{j\omega t} = \frac{L}{R_1+R_2} \frac{d}{dt}\left(\hat{V}_{out} e^{j\omega t}\right) + \hat{V}_{out} e^{j\omega t}$$

$$\frac{R_1 R_2}{R_1+R_2} I_{in} e^{j\omega t} = \frac{L}{R_1+R_2} e^{j\omega t} (j\omega \hat{V}_{out}) + \hat{V}_{out} e^{j\omega t}$$

$$\hat{V}_{out}\left(1 + j\omega \frac{L}{R_1+R_2}\right) = \frac{R_1 R_2}{R_1+R_2} I_{in}$$

$$\hat{V}_{out} = \frac{\frac{R_1 R_2}{R_1+R_2} I_{in}}{1 + j\omega \frac{L}{R_1+R_2}}$$

$$\hat{V}_{out} = \frac{R_1 R_2}{R_1+R_2+j\omega L} I_{in}$$

# Problem Set 10 Solutions

P10.1 (c)

$$\hat{V}_{out} = \frac{R_1 R_2 (R_1 + R_2 - j\omega L)}{(R_1+R_2)^2 + \omega^2 L^2} I_{in}$$

$$= \frac{R_1 R_2}{(R_1+R_2)^2 + \omega^2 L^2} \sqrt{(R_1+R_2)^2 + \omega^2 L^2}\, e^{-j \tan^{-1}\left[\frac{\omega L}{R_1+R_2}\right]} \cdot I_{in}$$

$$v_{OUT}(t) = \mathrm{Re}\{\hat{V}_{out} e^{j\omega t}\} = \mathrm{Re}\{|\hat{V}_{out}| e^{j\angle \hat{V}_{out}} e^{j\omega t}\}$$
$$= \mathrm{Re}\{|\hat{V}_{out}| e^{j(\omega t + \angle \hat{V}_{out})}\}$$

$$|\hat{V}_{out}| = \frac{R_1 R_2}{\sqrt{(R_1+R_2)^2 + \omega^2 L^2}} I_{in} \qquad \angle \hat{V}_{out} = -\tan^{-1}\left(\frac{\omega L}{R_1+R_2}\right)$$

Therefore, $v_{out}(t) = |\hat{V}_{out}| \cos(\omega t + \angle \hat{V}_{out})$

$$V_{out} = |\hat{V}_{out}| = \frac{R_1 R_2}{\sqrt{(R_1+R_2)^2 + \omega^2 L^2}} I_{in}$$

$$\phi = \angle \hat{V}_{out} = -\tan^{-1} \frac{\omega L}{R_1 + R_2}$$

(D)

$$\frac{V_{out}}{I_{in}} = \frac{R_1 R_2}{\sqrt{(R_1+R_2)^2 + \omega^2 L^2}}$$

$$= \frac{R_1 R_2}{R_1 + R_2} \cdot \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^2}} \qquad \omega_c = \frac{R_1 + R_2}{L}$$

$$\phi = -\tan^{-1}\left(\frac{\omega L}{R_1+R_2}\right) = -\tan^{-1}\left(\frac{\omega}{\omega_c}\right)$$

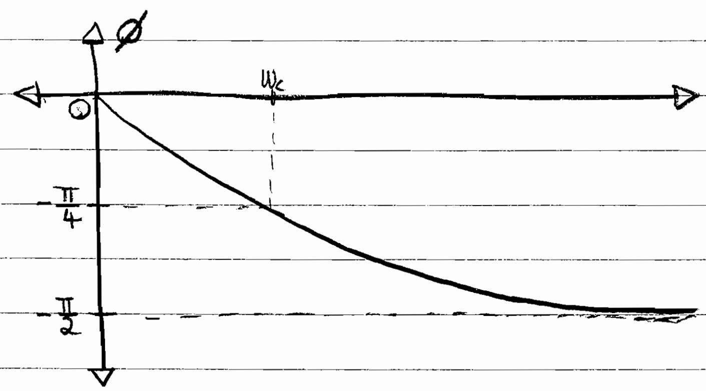
$\omega \ll \omega_c$ $\quad \frac{V_{out}}{I_{in}} \rightarrow \frac{R_1 R_2}{R_1 + R_2}$ $\quad \phi \rightarrow 0$

$\omega = \omega_c$ $\quad \frac{V_{out}}{I_{in}} = \frac{1}{\sqrt{2}} \cdot \frac{R_1 R_2}{R_1 + R_2}$ $\quad \phi = -\frac{\pi}{4}$

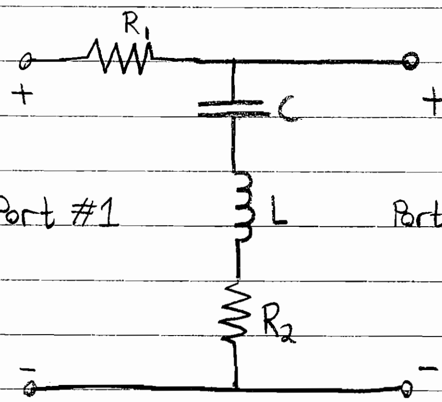
$\omega \gg \omega_c$ $\quad \frac{V_{out}}{I_{in}} \rightarrow \frac{R_1 R_2}{\omega L}$ $\quad \phi \rightarrow -\frac{\pi}{2}$

## Problem Set 10 Solutions

P.10.1 (D) Continued

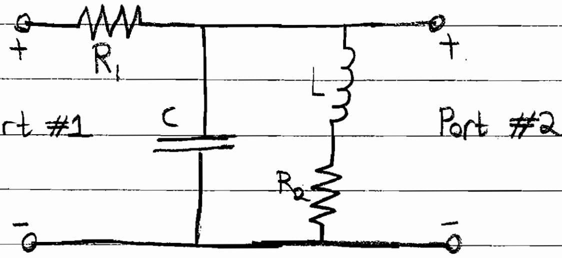

P.10.2 (A)




Network A

$$Z_{in,A} = R_1 + \frac{1}{j\omega C} + j\omega L + R_2$$

$$Z_{in,A} = \frac{(R_1+R_2)j\omega C - \omega^2 LC + 1}{j\omega C}$$




$$Z_{in,B} = R_1 + \frac{\frac{1}{j\omega C}(R_2 + j\omega L)}{R_2 + j\omega L + \frac{1}{j\omega C}}$$

$$Z_{in,B} = R_1 + \frac{R_2 + j\omega L}{1 + (R_2 + j\omega L)j\omega C}$$

$$= \frac{R_1 + R_2 - \omega^2 L R_1 C + j\omega(R_1 R_2 C + L)}{1 - \omega^2 LC + j\omega R_2 C}$$

Problem Set 10 Solutions

P.10.2 (B)

Port #1 $\rightarrow$ $v_1(t) = V_1 \cos(\omega t) \Rightarrow \tilde{v}_1 = V_1 e^{j\omega t}$

Port #2 $\rightarrow$ Open

Network A:

$$I_{in,A} = \frac{V_1}{Z_{in,A}} = \frac{V_1 j\omega C}{(R_1+R_2)j\omega C - \omega^2 LC + 1}$$

$$I_{in,A} = |I_{in,A}| e^{j\phi} \qquad \text{where} \quad \phi = \angle I_{in,A}$$

$$|I_{in,A}| = \frac{V_1 \omega C}{\sqrt{(1-\omega^2 LC)^2 + (\omega C(R_1+R_2))^2}}$$

$$\phi_{1,A} = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega C(R_1+R_2)}{1-\omega^2 LC}\right)$$

So $\quad i_{in,A}(t) = |I_{in,A}| \cos(\omega t + \phi_{1,A})$

Network B:

$$I_{in,B} = \frac{V_1}{Z_{in,B}} = \frac{V_1(1-\omega^2 LC + j\omega R_2 C)}{R_1+R_2-\omega^2 LR_1C + j\omega(R_1R_2C+L)}$$

$$|I_{in,B}| = \frac{V_1\sqrt{(1-\omega^2LC)^2+(\omega R_2C)^2}}{\sqrt{R_1+R_2-\omega^2LR_1C+\omega^2(R_1R_2C+L)^2}}$$

$$\phi_{1,B} = \tan^{-1}\left(\frac{\omega R_2 C}{1-\omega^2 LC}\right) - \tan^{-1}\left(\frac{\omega(R_1R_2C+L)}{R_1+R_2-\omega^2 LR_1C}\right)$$

So $\quad i_{in,B}(t) = |I_{in,B}| \cos(\omega t + \phi_{1,B})$

(C) Voltage at Port #2 is given by voltage divider:

Network A:

$$V_{2,A} = V_1 \frac{\frac{1}{j\omega C} + j\omega L + R_2}{R_1 + R_2 + \frac{1}{j\omega C} + j\omega L} = \frac{V_1(1-\omega^2LC + j\omega R_2 C)}{1-\omega^2 LC + j\omega(R_1+R_2)C}$$

$$|V_{2,A}| = \frac{V_1\sqrt{(1-\omega^2LC)^2+(\omega R_2C)^2}}{\sqrt{(1-\omega^2LC)^2+(\omega(R_1+R_2)C)^2}}$$

Prolem Set 10 Solutions

P.10.2 (C) Continued

$$\phi_{2,A} = \tan^{-1}\left(\frac{\omega R_2 C}{1-\omega^2 LC}\right) - \tan^{-1}\left(\frac{\omega(R_1+R_2)C}{1-\omega^2 LC}\right)$$

So $$v_{2,A}(t) = |V_{2,A}|\cos(\omega t + \phi_{2,A})$$

<u>Network B:</u>

$$V_{2,B} = \frac{V_1 \dfrac{\frac{1}{j\omega C}(j\omega L + R_2)}{\frac{1}{j\omega C} + R_2 + j\omega L}}{R_1 + \dfrac{\frac{1}{j\omega C}(j\omega L + R_2)}{\frac{1}{j\omega C} + R_2 + j\omega L}} = \frac{V_1}{1 + R_1 \dfrac{\frac{1}{j\omega C} + R_2 + j\omega L}{\frac{1}{j\omega C}(j\omega L + R_2)}}$$

$$V_{2,B} = \frac{V_1}{1 + \dfrac{R_1(1-\omega^2 LC + j\omega R_2 C)}{j\omega L + R_2}} = \frac{V_1(j\omega L + R_2)}{j\omega L + R_2 + R_1 - R_1\omega^2 LC + j\omega R_1 R_2 C}$$

$$|V_{2,B}| = \frac{V_1\sqrt{(\omega L)^2 + R^2}}{\sqrt{[R_2 + R_1(1-\omega^2 LC)]^2 + \omega^2(L + R_1 R_2 C)^2}}$$

$$\phi_{2,B} = \tan^{-1}\left(\frac{\omega L}{R_2}\right) - \tan^{-1}\left(\frac{\omega(L + R_1 R_2 C)}{R_2 + R_1(1-\omega^2 LC)}\right)$$

So $$v_{2,B}(t) = |V_{2,B}|\cos(\omega t + \phi_{2,B})$$

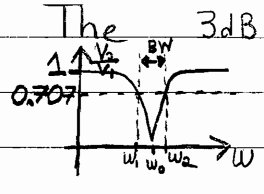
(D) Run Network A in Weblab. A screen shot of Weblab is shown on the next page which includes the magnitude and phase plots. The plots show the "notch" frequency is $f_n = 1\,\text{kHz}$. Since $\omega_n = 2\pi f_n = \frac{1}{\sqrt{LC}}$ we can solve for

$$C = \frac{1}{L(2\pi f_n)^2} = \frac{1}{(98\times10^{-3})[(2\pi)(1\times10^3)]^2} = 2.58\times10^{-7} = 258\,\text{nF}$$

Also, we know that for $\omega = \omega_n = \frac{1}{\sqrt{LC}}$

$$\frac{V_2}{V_1} = \frac{\omega R_2 C}{\omega(R_1+R_2)C} = \frac{R_2}{R_1+R_2} \simeq 0.1981$$ (From Weblab)

The 3dB frequencies (where gain drops to $\frac{1}{\sqrt{2}} = 0.707$):




$BW = \omega_2 - \omega_1$

From Weblab:

$f_1 \approx 660$ Hz $\qquad f_2 \approx 1747$ Hz

$\omega_1 = 4147$ rad/s $\qquad \omega_2 \approx 10{,}977$ rad/s

# Problem Set 10 solutions

P10.2 (D) Therefore:

$$\frac{V_2}{V_1} = \frac{\sqrt{(1-\omega_1^2 LC)^2 + (\omega_1 R_2 C)^2}}{\sqrt{(1-\omega_1^2 LC)^2 + (\omega_1 C(R_1+R_2)C)^2}} = \frac{1}{\sqrt{2}}$$

Let $a = (1-\omega_1^2 LC)^2 = 0.320$, $b = \frac{1}{0.1981} = 5.048$ $\quad \gamma = (\omega_1 C)^2$

$\Rightarrow \gamma = 1.145\times10^{-6}$

So $$\frac{a+\gamma R_2^2}{a+\gamma b^2 R_2^2} = \frac{1}{2}$$

$$\Rightarrow 2a + \gamma R_2^2 = a + \gamma b^2 R_2$$

$$\gamma R_2^2 (b^2-1) = a$$

$$\Rightarrow R_2 = \sqrt{\frac{a}{b^2-1}\cdot\frac{1}{\gamma}} = 106.8\,\Omega$$

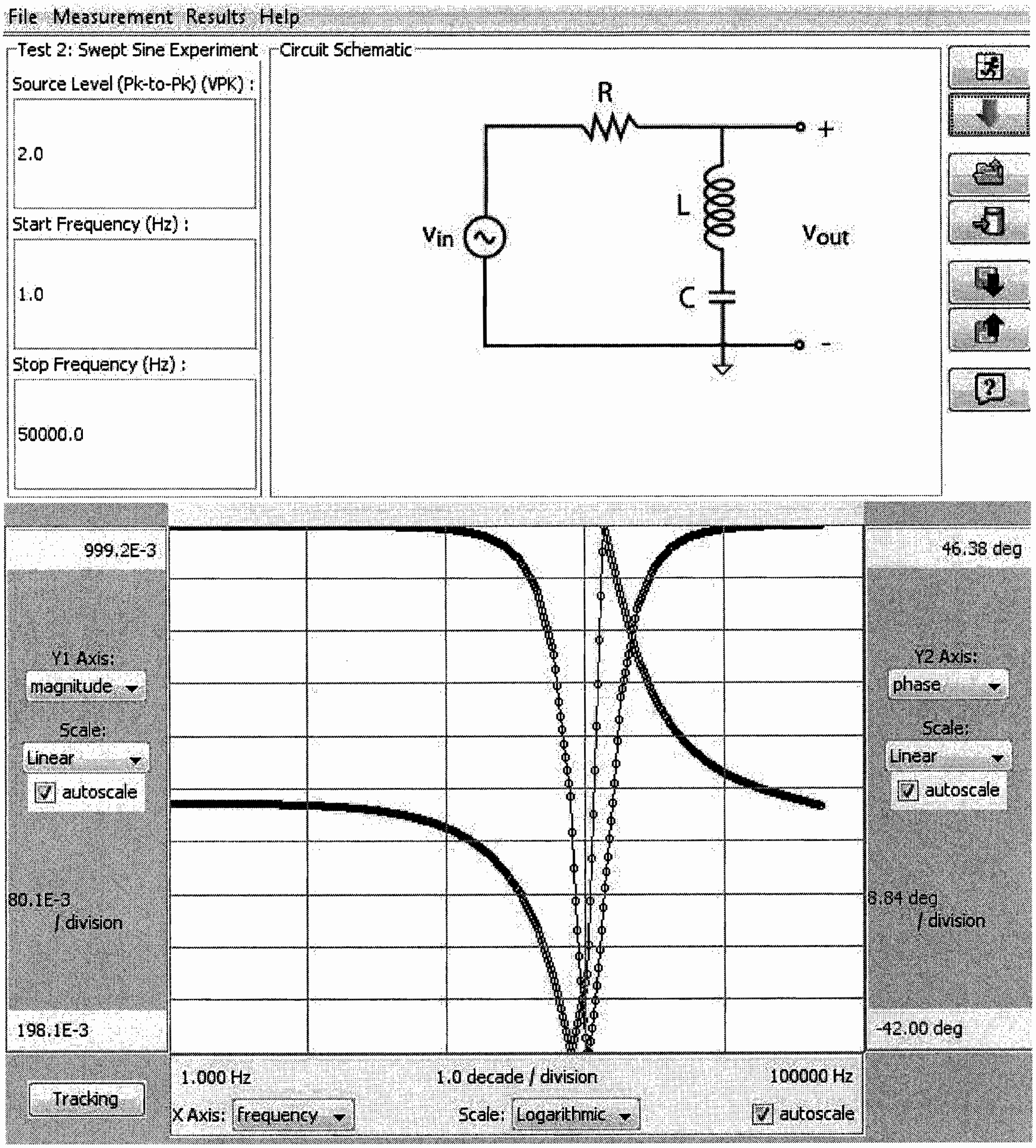
Given:

$$\frac{R_2}{R_1+R_2} = \frac{1}{b}$$

$$bR_2 = R_1 + R_2$$

$$R_1 = (b-1)\,R_2$$

$$R_1 = 432.3\,\Omega$$


File Measurement Results Help
Test 2: Swept Sine Experiment
Source Level (Pk-to-Pk) (VPK) :
2.0
Start Frequency (Hz) :
1.0
Stop Frequency (Hz) :
50000.0
Circuit Schematic
R
L
C
Vin
Vout
+
-
999.2E-3
Y1 Axis:
magnitude
Scale:
Linear
autoscale
80.1E-3
/ division
198.1E-3
46.38 deg
Y2 Axis:
phase
Scale:
Linear
autoscale
8.84 deg
/ division
-42.00 deg
1.000 Hz
1.0 decade / division
100000 Hz
X Axis: frequency
Scale: Logarithmic
autoscale
Tracking

# Problem Set 10 Solutions

P10.3 (A)

$$I(t) = I\cos(\omega t) = \mathrm{Re}\{I e^{j\omega t}\}$$

$$V(j\omega) = I\left(j\omega L // \frac{1}{j\omega C} // R\right) = I\left(\frac{j\omega LR}{j\omega L + R} // \frac{1}{j\omega C}\right)$$

$$= \frac{IL\frac{R}{C}}{(j\omega L + R)\left(\frac{j\omega LR}{j\omega L+R} + \frac{1}{j\omega C}\right)}$$

$$= \frac{IL\frac{R}{C}}{j\omega RL + \frac{j\omega L + R}{j\omega C}}$$

$$= \frac{Ij\omega LR}{R(1-\omega^2 LC) + j\omega L}$$

$$V(\omega) = \frac{I\omega LR}{\sqrt{R^2(1-\omega^2 LC)^2 + (\omega L)^2}}$$

$$\phi(\omega) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L}{R(1-\omega^2 LC)}\right) = \tan^{-1}\left(\frac{R(1-\omega^2 LC)}{\omega L}\right)$$

(B) max $V(\omega) \rightarrow$ max $Z$

$Z$ is max when $Z_L // Z_C \rightarrow \infty$, when $\omega = \frac{1}{\sqrt{LC}}$

So impedance $Z$ is just $R$

(C) $L = 470\mu H$ Range of $C$?

Max $V/I \rightarrow$ max $Z \rightarrow \omega = \frac{1}{\sqrt{LC}}$ from part (b)

Frequency range $520\ kHz < f < 1610\ kHz$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \rightarrow C = \frac{1}{(2\pi f)^2 L}$$

Minimum $C$: $f = 1610\ kHz$

$C = 20.8\ pF$

Maximum $C$: $f = 520\ kHz$

$C = 199.3\ pF$

# Problem Set 10 Solutions

P.10.3 (D) When $f = 1\,MHz$,

$$\frac{V(1MHz \pm 5kHz)}{V(1MHz)} \approx 0.25$$

$\omega_0 = \frac{1}{\sqrt{LC}}$ so $Z = R$

Thus $V = V_0 = IR$ @ $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$ $(f = 1MHz)$

As $\omega$ moves away from $\omega_0$, $Z$ decreases, decreasing $V_0$

For $f = 1MHz + 5KHz = 1.005\,MHz$, $\omega_1 = 1.005\,\omega_0 = \frac{1.005}{\sqrt{LC}} = \frac{\gamma}{\sqrt{LC}} = \gamma\omega_0$

$$V(\gamma\omega_0) = 0.25\,IR$$

$$V(\gamma\omega_0) = \frac{I\,\omega_1 L R}{\sqrt{R^2(1-\omega_1^2 LC)^2 + (\omega_1 L)^2}} = 0.25\,IR$$

$$\therefore \quad \frac{\gamma\sqrt{\frac{L}{C}}}{\sqrt{R^2(1-\gamma^2)^2 + \gamma^2\frac{L}{C}}} = 0.25$$

$$\frac{\gamma^2}{\frac{C}{L}R^2(1-\gamma^2)^2 + \gamma^2} = (0.25)^2$$

$$\frac{C}{L}R^2(1-\gamma^2)^2 + \gamma^2 = \frac{\gamma^2}{(0.25)^2}$$

$$R = \sqrt{\frac{\left[\left(\frac{\gamma}{0.25}\right)^2 - \gamma^2\right]L}{(1-\gamma^2)^2 C}}$$

$$R = \sqrt{\frac{15\gamma^2 L}{(1-\gamma^2)^2 C}}$$

$L = 470\,\mu H$

$C = \frac{1}{(2\pi f)^2 L} = 53.9\,pF$

For $\gamma = 1.005$ $\quad R = 1{,}147\,K\Omega$

$\gamma = 0.995$ $\quad R = 1{,}141\,k\Omega$

Choose larger $R$ for better attenuation @ $1MHz \pm 5KHz$

$R = 1{,}147\,K\Omega$

(E) Find $\frac{V(1MHz \pm 10KHz)}{V(1MHz)}$ $\quad \therefore$ Let $\gamma = 1.01, 0.99$

$$\frac{V(\gamma\omega_0)}{V(\omega_0)} = \frac{I\frac{\gamma}{\sqrt{LC}}LR}{\sqrt{R^2(1-\gamma^2)^2 + \gamma^2\frac{L}{C}}} \cdot \frac{1}{IR} = \frac{\gamma}{\sqrt{\frac{C}{L}R^2(1-\gamma^2)^2 + \gamma^2}}$$

For $\gamma = 1.01$, $\frac{V(1.01\omega_0)}{V(\omega_0)} = 0.128$ $\qquad$ For $\gamma = 0.99$, $\frac{V(0.99\omega_0)}{V(\omega_0)} = 0.127$

P.10.3 (E) Continued

$f = 1\,\text{MHz}$:

$$Q = \frac{\omega_0}{2\alpha} = \frac{1}{\sqrt{LC}} \cdot \frac{1}{2} \cdot \frac{2RC}{1} = R\sqrt{\frac{C}{L}} = 388$$

(F) Here the transient, which decays as $e^{-\alpha t}$, needs to be considered, where $\alpha = \frac{1}{2RC}$. The "$\tau$" of this is $\frac{1}{\alpha} = 2RC$, which is $124\,\mu s$. In general we can say that the transient has nearly died out after $\sim 5\tau$ (99% decay)

$$5\tau = 620\,\mu s$$