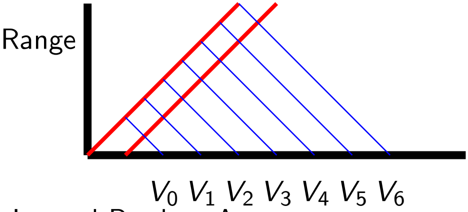
Basic Radar 3.3: Spectral Estimation

Roger H. Varney

[1] Center for Geospace Studies
SRI International

July, 2021

Lagged Product Arrays (LPA)



Lagged Product Array:

$$L_\ell^i \equiv \left\langle V^*_{i-\left\lfloor \frac{\ell}{2} \right\rfloor} V_{i+\left\lfloor \frac{\ell}{2} \right\rfloor + (\ell \bmod 2)} \right\rangle$$

$$L_0^i = \langle V_i^* V_i \rangle$$

$$L_1^i = \langle V_i^* V_{i+1} \rangle$$

$$L_2^i = \left\langle V_{i-1}^* V_{i+1} \right\rangle$$

$$L_3^i = \left\langle V_{i-1}^* V_{i+2} \right\rangle$$

etc.

The set of voltage samples from N ranges after a single pulse could be completely characterized by the $N \times N$ covariance matrix. However,

- Covariance matrices are always conjugate symmetric (Hermitian).
- Only samples within 1 pulse length of each other will have non-zero covariance.

For an experiment with 16 samples per pulse length we only need a $N \times 16$ array of covariances.

LPA Estimators

This definition

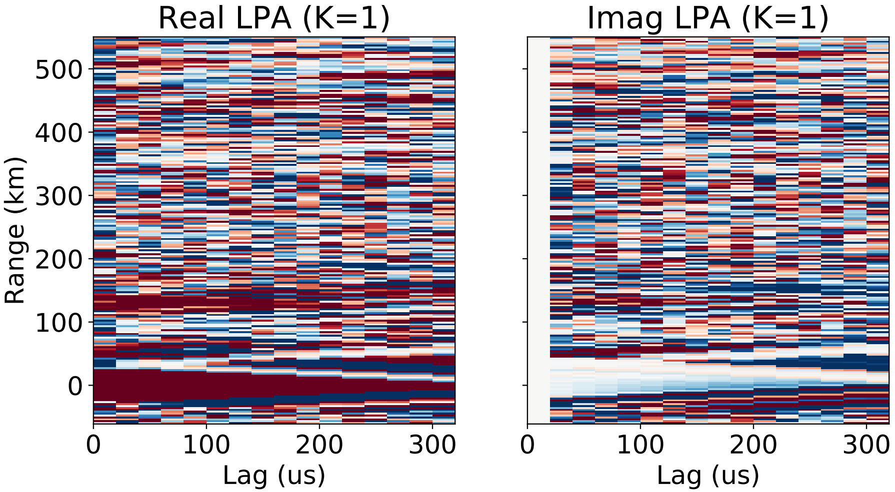
$$L_\ell^i \equiv \left\langle V^*_{i-\lfloor \frac{\ell}{2} \rfloor} V_{i+\lfloor \frac{\ell}{2} \rfloor + (\ell \bmod 2)} \right\rangle$$

refers to the *expected value* of the products of samples.
Can be estimated by taking products samples and averaging over many pulses

$$\hat{L}_\ell^i = \frac{1}{K} \sum_{k=0}^{K-1} V^*_{i-\lfloor \frac{\ell}{2} \rfloor} V_{i+\lfloor \frac{\ell}{2} \rfloor + (\ell \bmod 2)}$$

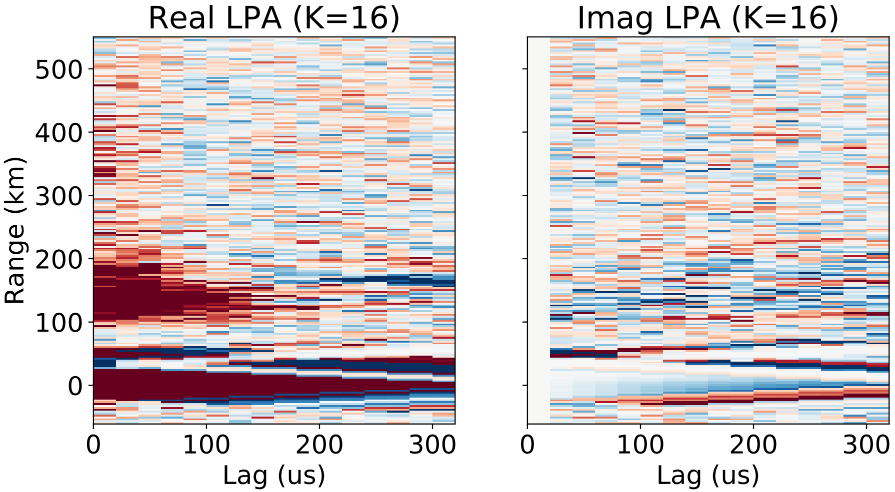
By virtue of the central limit theorem, these estimators will be Gaussian random variables with variances that decrease as $1/K$ (error decreases as $1/\sqrt{K}$).

LPA Estimator from 1 Pulse



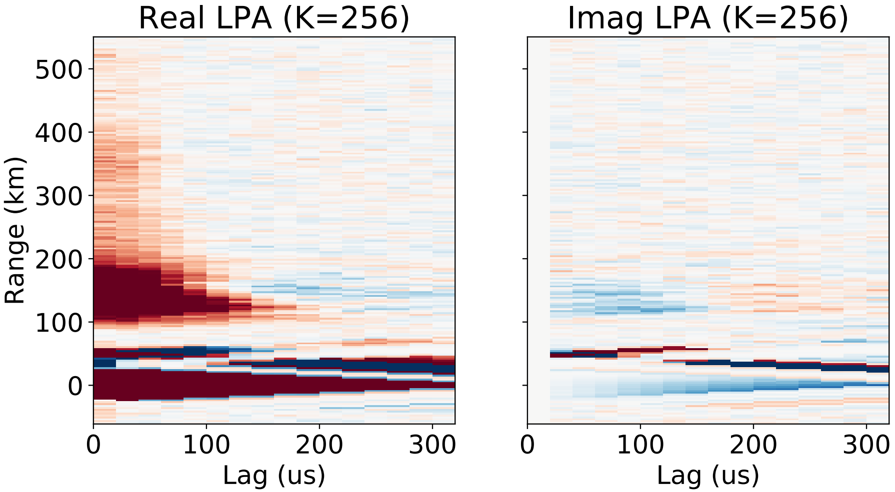
Note the Noise ACF has been subtracted before plotting.

LPA Estimator from 16 Pulses



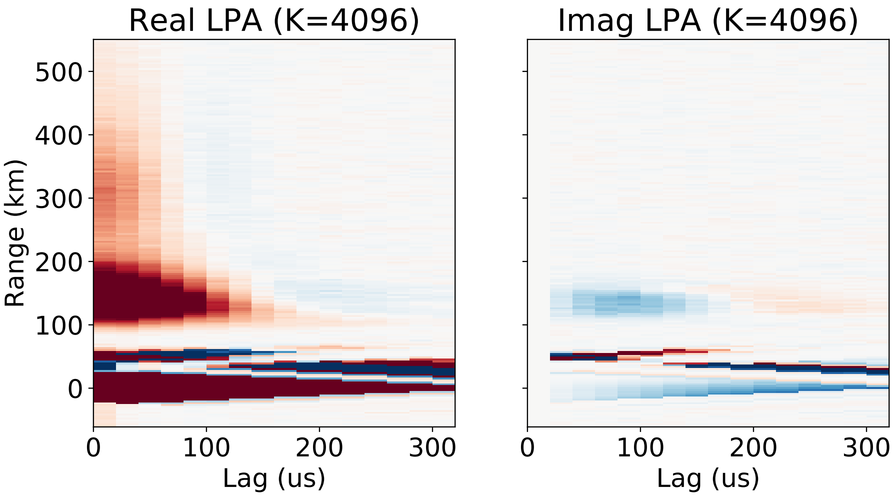
Note the Noise ACF has been subtracted before plotting.

LPA Estimator from 256 Pulses



Note the Noise ACF has been subtracted before plotting.

LPA Estimator from 4096 Pulses



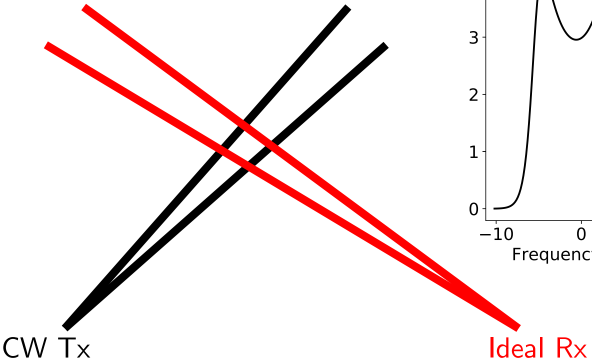
Note the Noise ACF has been subtracted before plotting.

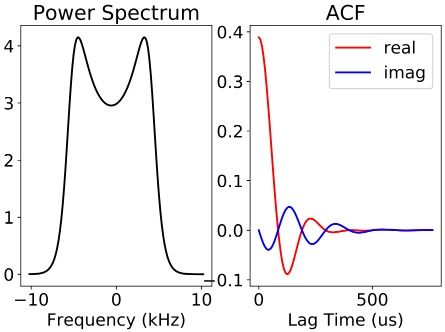
Forward Model for ISR LPA

Goal: Map ionospheric physical parameters ($N_e(R)$, $T_e(R)$, $T_i(R)$, $u_{\mathrm{los}}(R)$) to the LPA the radar is expected to measure L_ℓ^i.

1. Physics portion (ISR theory)
2. Engineering portion (Ambiguity function theory)
3. Noise background

Physics Portion

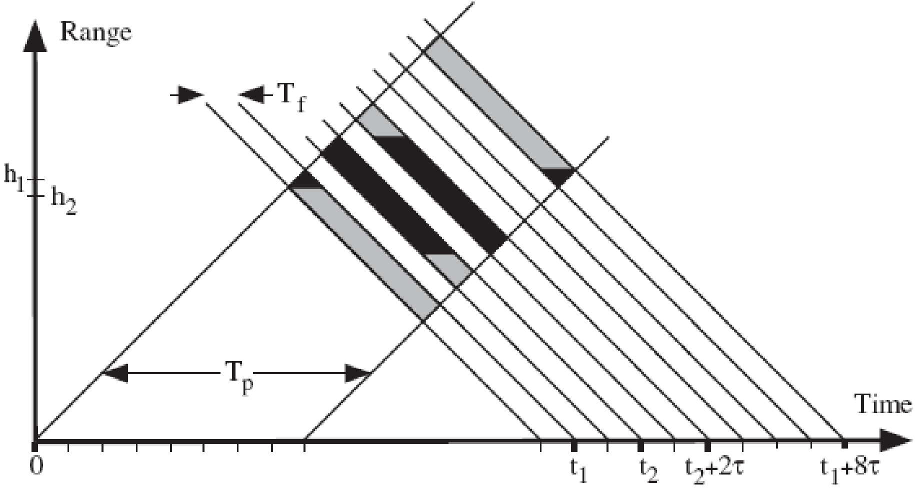




ISR theory gives the PSD and ACF of the received voltages as a function of N_e, T_e, T_i, and u_{los} in the overlap volume.

Engineering Portion

LPA Mixes Range and Time Information



Information smeared in range over the dark overlap regions.

Ambiguity Functions

Measured lag-products:

$$\langle V(t_{s2})\, V^*(t_{s1})\rangle = \int d\tau dr\, A(r,\tau)\, W_{t_{s1},t_{s2}}(\tau, r)$$

The measured lag-product is the ACF from the physics portion $A(r,\tau)$ blurred the the **range-lag ambiguity function** $W_{t_{s1},t_{s2}}(\tau, r)$

The ambiguity function is a function of the transmit waveform $s(t)$ and the receiver impulse response $h(t)$

$$W_{t_s}(t, r) \equiv s\left(t - \frac{2r}{c}\right) h^*(t_s - t)$$

$$W_{t_{s1},t_{s2}}(\tau, r) = \int dt\, W_{ts2}(t+\tau, r)\, W^*_{ts1}(t, r)$$

Full Forward Model Including External Noise

$$\langle V(t_{s2}) V^*(t_{s1}) \rangle = \int d\tau dr\, A(r,\tau) W_{t_{s1},t_{s2}}(\tau, r) + R_N(t_{s2} - t_{s1})$$

Measured elements of the LPA are determined by

- Parameters of the ionosphere (unknown)
- Engineering details of radar experiment (known)
- ACF of external noise (estimate from long ranges)

Goal of fitting is to go backwards from measurements to parameters of the ionosphere.