Basic Radar 3.1: Probability Theory for Incoherent Scatter Radar

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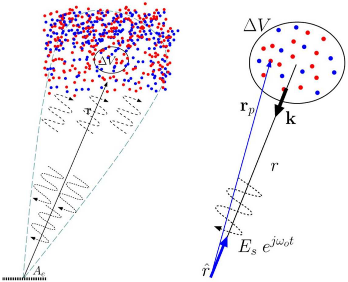
July, 2021

The Need for Statistical Descriptions of ISR Signals

If I knew the positions of every single electron in the scattering volume, I would know the received voltage exactly:



Exact expression for scattered electric field as a superposition of Thomson scatterers:

$$E_s = -\frac{r_e}{r} E_0 \sum_{p=1}^{N_0 \Delta V} e^{j\mathbf{k}\cdot\mathbf{r}_p}$$

ISR theory predicts statistical aspects of the scattered signal:

Scattered Power: $\left\langle |E_s|^2 \right\rangle$ Autocorrelation Function: $\langle E_s(t) E_s^*(t-\tau) \rangle$

These statistical properties are functions of macroscopic properties of the plasma: N_e, T_e, T_i, u_{los}.

Random Variables

A **random variable** is a variable whose numerical value depends on the outcome of a probabilistic phenomenon.

Probability Density Function:

$$P\left(x_1 < X < x_2\right) = \int_{x_1}^{x_2} p_X(x)\, dx$$

Expected Values:

$$E\left\{g\left(X\right)\right\} = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx$$

Mean:

$$Mean\left\{X\right\} = E\left\{X\right\} = \bar{X}$$

Variance:

$$Var\left\{X\right\} = E\left\{\left(X - E\left\{X\right\}\right)^2\right\} = E\left\{X^2\right\} - \left(E\left\{X\right\}\right)^2$$

Collections of Random Variables

Multiple RVs must be described by joint-PDFs:

$$P\left(x_0 < X < x_1 \cup y_0 < Y < y_1\right) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} p_{XY}\left(x, y\right)\, dydx$$

If X and Y are **independent**:

$$p_{XY}(x,y) = p_X(x)p_Y(y) \qquad p_{X|Y}(x|y) = p_X(x)$$

Relationships between RVs are defined through covariances:

$$Cov\left\{X, Y\right\} = E\left\{\left(X - E\{X\}\right)\left(Y - E\{Y\}\right)\right\}$$

Uncorrelated RVs have $Cov\{X, Y\} = 0$
Independent RVs are uncorrelated, but uncorrelated RVs are not necessarily independent.

Random Vectors and Covariance Matrices

Column vector of random variables

$$\mathsf{X} = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{pmatrix}$$

Covariance matrix of a random vector

$$K_{\mathsf{X}} = \mathrm{Cov}\left\{\mathsf{X}\right\} = E\{\left(\mathsf{X} - \bar{\mathsf{X}}\right)\left(\mathsf{X} - \bar{\mathsf{X}}\right)^T\}$$

Cross-covariance of two random vectors

$$K_{\mathsf{XY}} = \mathrm{Cov}\left\{\mathsf{X}, \mathsf{Y}\right\} = E\{\left(\mathsf{X} - \bar{\mathsf{X}}\right)\left(\mathsf{Y} - \bar{\mathsf{Y}}\right)^T\}$$

$$K_X = \begin{pmatrix} \mathrm{Var}\left\{X_0\right\} & \mathrm{Cov}\left\{X_0, X_1\right\} & \cdots & \mathrm{Cov}\left\{X_0, X_{N-1}\right\} \\ \mathrm{Cov}\left\{X_1, X_0\right\} & \mathrm{Var}\left\{X_1\right\} & \cdots & \mathrm{Cov}\left\{X_1, X_{N-1}\right\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}\left\{X_{N-1}, X_0\right\} & \mathrm{Cov}\left\{X_{N-1}, X_1\right\} & \cdots & \mathrm{Var}\left\{X_{N-1}\right\} \end{pmatrix}$$

Properties:

$$\mathsf{Y} = A\mathsf{X} + \mathsf{b} \rightarrow \bar{\mathsf{Y}} = A\bar{\mathsf{X}} + \mathsf{b} \qquad K_Y = AK_XA^T$$

Gaussian Distribution

A Gaussian random variable X has the following probability density function (Normal Distribution):

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x-\mu}{2\sigma^2}\right\}$$

$$E\{X\} = \mu \qquad Var\{X\} = \sigma^2$$

$$E\left\{(X-\mu)^4\right\} = 3\sigma^4$$

A jointly-Gaussian vector of random variables $\mathrm{X} = [X_0, X_1, X_2, \cdots, X_{N-1}]^T$ has the joint pdf:

$$p(\mathrm{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} |K|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}[\mathrm{x}-\mu]^T K^{-1} [\mathrm{x}-\mu]\right\}$$

$$E\{\mathrm{X}\} = \mu$$

$$Cov\{\mathrm{X}\} = E\left\{[\mathrm{X}-\mu][\mathrm{X}-\mu]^T\right\} = K$$

Central Limit Theorem

Given a set of finite-variance, independent and identically distributed RV, $[X_0, X_1, \cdots, X_{K-1}]$, the distribution function of the average:

$$\hat{X} = \frac{1}{K} \sum_{n=0}^{K-1} X_n$$

will asymptotically approach a Gaussian distribution as K increases.

$$E\left\{\hat{X}\right\} = E\left\{X_n\right\} \qquad Var\left\{\hat{X}\right\} = \frac{1}{K} Var\left\{X_n\right\}$$

This is an amazingly useful theorem:

- Only the mean and variances of the intermediate quantities need to be calculated to predict the distribution of the final averaged result.
- Distribution functions of intermediate quantities do not need to be calculated in detail since the final averaged result will just be Gaussian.

Complex-valued Random Variables

A vector of complex-valued random variables can be written as two vectors of real-valued random variables representing the real and imaginary parts.

$$\mathsf{X} = \mathsf{X}_R + j\mathsf{X}_I$$

$$\text{Mean:}\bar{\mathsf{X}} = E\left\{\mathsf{X}\right\} = E\left\{\mathsf{X}_\mathsf{R}\right\} + jE\left\{\mathsf{X}_\mathsf{I}\right\}$$

$$\text{Covariance:}K_X = E\left\{\left(\mathsf{X}-\bar{\mathsf{X}}\right)\left(\mathsf{X}-\bar{\mathsf{X}}\right)^H\right\}$$

$$\text{Pseudo-Covariance:}J_X = E\left\{\left(\mathsf{X}-\bar{\mathsf{X}}\right)\left(\mathsf{X}-\bar{\mathsf{X}}\right)^T\right\}$$

The covariance and cross-covariance matrices of the real vectors are related to the covariance and pseudo-covariance of the complex vector by

$$\mathrm{Cov}\left\{\mathsf{X}_R\right\} = \frac{1}{2}\Re K_X + \frac{1}{2}\Re J_X \qquad \mathrm{Cov}\left\{\mathsf{X}_I\right\} = \frac{1}{2}\Re K_X - \frac{1}{2}\Re J_X$$

$$\mathrm{Cov}\left\{\mathsf{X}_I, \mathsf{X}_R\right\} = \frac{1}{2}\Im K_X + \frac{1}{2}\Im J_X \qquad \mathrm{Cov}\left\{\mathsf{X}_R, \mathsf{X}_I\right\} = -\frac{1}{2}\Im K_X + \frac{1}{2}\Im J_X$$

Describing ISR Voltages

ISR signals are complex valued, zero mean, and random phase.

$$V = V_R + jV_I \qquad E\left\{V_R\right\} = E\left\{V_I\right\} = 0$$

$$E\left\{VV^*\right\} = \sigma^2 \qquad E\left\{V_R V_I\right\} = 0 \qquad \mathrm{Cov}\left\{\begin{pmatrix} V_R \\ V_I \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

When we talk about correlations between ISR signals

$$E\left\{V_1 V_1^*\right\} = \sigma_1^2 \quad E\left\{V_2 V_2^*\right\} = \sigma_2^2$$

$$E\left\{V_1 V_2^*\right\} = \rho = \rho_R + j\rho_I$$

What we really mean is

$$V_1 = V_{1R} + jV_{1I} \qquad V_2 = V_{2R} + jV_{2I}$$

$$\mathrm{Cov}\left\{\begin{pmatrix} V_{1R} \\ V_{1I} \\ V_{2R} \\ V_{2I} \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} \sigma_1^2 & 0 & \rho_R & -\rho_I \\ 0 & \sigma_1^2 & \rho_I & \rho_R \\ \rho_R & \rho_I & \sigma_2^2 & 0 \\ -\rho_I & \rho_R & 0 & \sigma_2^2 \end{pmatrix}$$

Stochastic Processes

- Stochastic Process (aka Random Process):
  $V(t)$ where value at every time is a random variable
- Gaussian Stochastic Process:
  - PDF of each $V(t)$ is a Gaussian distribution (aka normal distribution)
  - Joint PDF of any subset of samples of $V(t)$ is a jointly Gaussian distribution (aka Multivariate Normal Distribution)
- Moments of a Stochastic Process:
  - Mean: $\bar{V}(t) = E\left\{V(t)\right\}$
  - Autocorrelation: $R_V\left(t, t-\tau\right) = E\left\{V(t)V^*(t-\tau)\right\}$
  - Autocovariance:
    $C_V\left(t, t-\tau\right) = E\left\{\left[V(t) - \bar{V}(t)\right]\left[V^*(t-\tau) - \bar{V}^*(t-\tau)\right]\right\}$
    $C_V\left(t, t-\tau\right) = R\left(t, t-\tau\right) - \bar{V}(t)\bar{V}^*(t-\tau)$
- (Wide Sense) Stationary Stochastic Process
  - $\bar{V}(t) = \bar{V}$ is independent of t
  - $R(t, t-\tau) = R(\tau)$ is independent of t

Power Spectra of Deterministic Signals

Given a signal $f(t)$ and its fourier transform $F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$, the power spectrum is:

$$
\begin{aligned}
S_F(\omega) = |F(\omega)|^2 &= F^*(\omega)F(\omega) \\
&= \mathcal{F}\left\{f(-t') * f(t')\right\} \\
&= \mathcal{F}\left\{\int_{-\infty}^{\infty} f(t')f(t'-t)\, dt'\right\}
\end{aligned}
$$

When you filter a signal:

$$
\begin{aligned}
g(t) &= h(t) * f(t) \\
G(\omega) &= H(\omega)F(\omega) \\
S_G(\omega) &= |H(\omega)|^2 \, S_F(\omega)
\end{aligned}
$$

Power Spectra of Stochastic Signals

Fourier transforms of stationary random processes do not exist.
Fourier transforms of ACFs will exist, and are the power spectra:

$$S_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau) e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} E\{V(t)V^*(t-\tau)\} e^{-j\omega\tau}\, d\tau$$

Properties:

- $S(\omega)$ is real and $S(\omega) \geq 0$
- Short correlation times $\leftrightarrow$ wide bandwidth and vice versa
- $\int_{-\infty}^{\infty} S_V(\omega)\, d\omega = R(0) = E\{|V|^2\}$ (total power)
- If U = h * V, $S_U(\omega) = |H(\omega)|^2\, S_V(\omega)$

Intuitive interpretation: $\int_{\omega_1}^{\omega_2} S_V(\omega)\, d\omega$ is the power in the frequency band from ω_1 to ω_2.

Building Blocks

- Random Variables (Mean, Variance)
- Random Vectors (Mean, Covariance Matrix)
- Complex Random Vectors (Mean, Complex Covariance Matrix, Complex Pseudo-Covariance Matrix)
- Stochastic Processes (Mean, Autocorrelation Function, Power Spectrum)