ISR Theory 3: Dispersion Relations

Roger H. Varney

[1] Center for Geospace Studies
SRI International

July, 2021

Dispersion Relations

- A **dispersion relation** is an algebraic equation of frequency ω and wavenumber $\mathbf{k}$.

$$\mathcal{D}(\omega, \mathbf{k}) = 0$$

- Dispersion relations are associated with sets of linear partial differential equations
- If $\omega, \mathbf{k}$ solve the dispersion relation, then there exists a set of fields of the form $\mathbf{E} = \mathbf{E}_0 e^{j\omega t - j\mathbf{k}\cdot\mathbf{x}}$ that solve the linear PDEs.

Example:

Maxwell's Equations in Vacuum

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$
$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B}$$

Vacuum Dispersion Relation

$$\omega = ck$$

High Frequency Waves in Cold Plasma

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$
$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \left[\nabla \times \mathbf{B} - \mu_0 \mathbf{J}\right]$$
$$\frac{\partial \mathbf{J}}{\partial t} = \frac{e^2}{m_e} N_e \mathbf{E}$$

Assume plane waves and solve for E

$$c^2 \frac{(\mathbf{k} \times \mathbf{E}) \times \mathbf{k}}{\omega^2} = \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right) \mathbf{E}$$
$$\omega_{pe}^2 \equiv \frac{e^2 N_e}{m_e \epsilon_0}$$

Dispersion relations:

- Electrostatic solution $\mathbf{k} \times \mathbf{E} = 0$

$$\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right) = 0$$
$$\omega^2 = \omega_{pe}^2$$

- Electromagnetic solution $(\mathbf{k} \times \mathbf{E}) \times \mathbf{k} = k^2 \mathbf{E}$

$$\underbrace{c^2 \frac{k^2}{\omega^2}}_{n^2} = \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)$$
$$\omega^2 = \omega_{pe}^2 + c^2 k^2$$

Phase Velocity and Group Velocity

Phase Velocity

- Velocity of a peak/trough
- $v_\phi = \frac{\omega}{k}$

Example

$$c^2 \frac{k^2}{\omega^2} = \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)$$

$$\frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}}$$

Group Velocity

- Velocity of a wave packet
- $v_g = \frac{d\omega}{dk}$

Example

$$\omega^2 = \omega_{pe}^2 + c^2 k^2$$

$$2\omega \frac{d\omega}{dk} = 2kc^2$$

$$\frac{d\omega}{dk} = \frac{k}{\omega} c^2 = c\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}$$

Dispersion Relations in Plasma Physics

Many different sets of PDEs used to describe waves in plasmas

- Electrostatic ($\nabla \times \mathrm{E} = 0$) vs Electromagnetic
- High frequency limit (only electrons move) vs low frequency limit
- Background magnetic field included
- Particles described as cold (fluid theory) or warm (kinetic theory)

Notable examples

- Appleton-Hartree equation
  - High frequency limit
  - Cold plasma (fluid)
  - Background magnetic field included
- Alfvén wave dispersion relations
  - Low frequency limit
  - Cold plasma (fluid) $\rightarrow$ ideal MHD
  - Background magnetic field included

Vlasov-Poisson System of Equations

- Warm plasma (kinetic)
- Electrostatic

$$\frac{\partial f_e}{\partial t} + \mathrm{v} \cdot \frac{\partial f_e}{\partial \mathrm{x}} - \frac{e}{m_e}\mathrm{E} \cdot \frac{\partial f_e}{\partial \mathrm{v}} = 0$$

$$\frac{\partial f_i}{\partial t} + \mathrm{v} \cdot \frac{\partial f_i}{\partial \mathrm{x}} + \frac{e}{m_i}\mathrm{E} \cdot \frac{\partial f_i}{\partial \mathrm{v}} = 0$$

$$\epsilon_0 \nabla \cdot \mathrm{E} = e\left(n_i - n_e\right)$$

$$n_e = \int d^3v\, f_e$$

$$n_i = \int d^3v\, f_i$$

Plasma as a Dielectric

Susceptibility relates fields E to material polarization P

$$\mathsf{P} = \epsilon_0 \chi \mathsf{E}$$
$$\rho_c = -\nabla \cdot \mathsf{P} = -\epsilon_0 \nabla \cdot (\chi \mathsf{E})$$
$$\rho_c = qn \to n = \frac{-\epsilon_0}{q} \nabla \cdot (\chi \mathsf{E})$$

Apply to Gauss' law

$$\epsilon_0 \nabla \cdot \mathsf{E} = e\,(n_i - n_e)$$
$$\epsilon_0 \nabla \cdot \mathsf{E} = e \left(\frac{-\epsilon_0}{e} \nabla \cdot (\chi_i \mathsf{E}) - \frac{\epsilon_0}{e} \nabla \cdot (\chi_e \mathsf{E}) \right)$$
$$\epsilon_0 \nabla \cdot \left[(1 + \chi_e + \chi_i)\, \mathsf{E} \right] = 0$$

Dispersion Relation

$$\underbrace{1 + \chi_e\,(\omega, \mathsf{k}) + \chi_i\,(\omega, \mathsf{k})}_{\epsilon} = 0$$

Electrostatic Plasma Waves

Approximate solutions to $\epsilon = 0$

Ion Acoustic Wave

$$\omega_r = k\sqrt{\frac{k_B}{m_i}\left(T_e + \gamma_i T_i\right)} \qquad \gamma_i \approx 3$$

$$\omega_i = \text{Strong function of } \frac{T_e}{T_i}$$

Langmuir Wave

$$\omega_r^2 = \omega_{pe}^2\left(1 + 3k^2\lambda_{De}^2\right)$$

$$\omega_i = \sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{k^3\lambda_{De}^3}\exp\left[-\left(1 + 3k^2\lambda_{De}^2\right)/2k^2\lambda_{De}^2\right]$$

Dressed Particle Theory

Superposition of test particles and field responses

$$n_e = n_{te} + \delta N_e = n_{te} - j\frac{\epsilon_0}{e}\chi_e \mathrm{k}\cdot\mathrm{E}$$

Gauss' Law

$$-j\epsilon_0 \mathrm{k}\cdot\mathrm{E} = e\left(n_i - n_e\right)$$

$$-j\epsilon_0 \mathrm{k}\cdot\mathrm{E} = e\left(n_{ti} + j\frac{\epsilon_0}{e}\chi_i \mathrm{k}\cdot\mathrm{E} - n_{te} + j\frac{\epsilon_0}{e}\chi_e \mathrm{k}\cdot\mathrm{E}\right)$$

$$-j\epsilon_0\left(1+\chi_e+\chi_i\right)\mathrm{k}\cdot\mathrm{E} = en_{ti} - en_{te}$$

Substituting back in

$$n_e = n_{te} - j\frac{\epsilon_0}{e}\chi_e\frac{en_{ti} - en_{te}}{-j\epsilon_0\left(1+\chi_e+\chi_i\right)}$$

$$= \frac{1+\chi_i}{1+\chi_e+\chi_i}n_{te} + \frac{\chi_e}{1+\chi_e+\chi_i}n_{ti}$$

ISR Spectrum

Power spectrum (note n_{te} and n_{ti} are uncorrelated)

$$\left\langle |n_e(\mathrm{k},\omega)|^2 \right\rangle = \frac{|1+\chi_i|^2}{|1+\chi_e+\chi_i|^2} \left\langle |n_{te}(\mathrm{k},\omega)|^2 \right\rangle + \frac{|\chi_e|^2}{|1+\chi_e+\chi_i|^2} \left\langle |n_{ti}(\mathrm{k},\omega)|^2 \right\rangle$$

Alternate Form

$$\left\langle |n_e(\mathrm{k},\omega)|^2 \right\rangle = \left|1-\frac{\chi_e}{\epsilon}\right|^2 \left\langle |n_{te}(\mathrm{k},\omega)|^2 \right\rangle + \left|\frac{\chi_e}{\epsilon}\right|^2 \left\langle |n_{ti}(\mathrm{k},\omega)|^2 \right\rangle$$

- Dispersion relation $\epsilon = 0 \Rightarrow$ small denominator $\Rightarrow$ peaks in spectrum
- Peaks where $\omega = k_b v_\phi$

Analogy to Filtered Noise

Can you hear the ocean in a seashell?

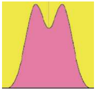
Resonant Filter

Ambient Noise $\longrightarrow$  $\longrightarrow$ Colored Noise (Ocean Sounds)

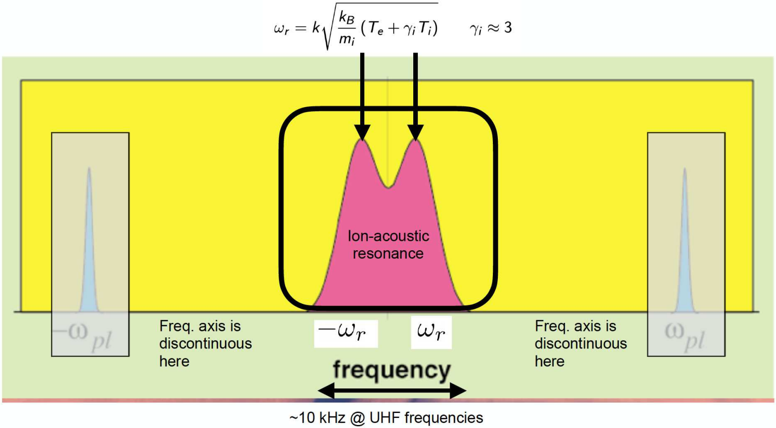
Analogy to ISR:

Thermal Motion $\longrightarrow$ Collective Effects $\longrightarrow$ ISR Spectrum

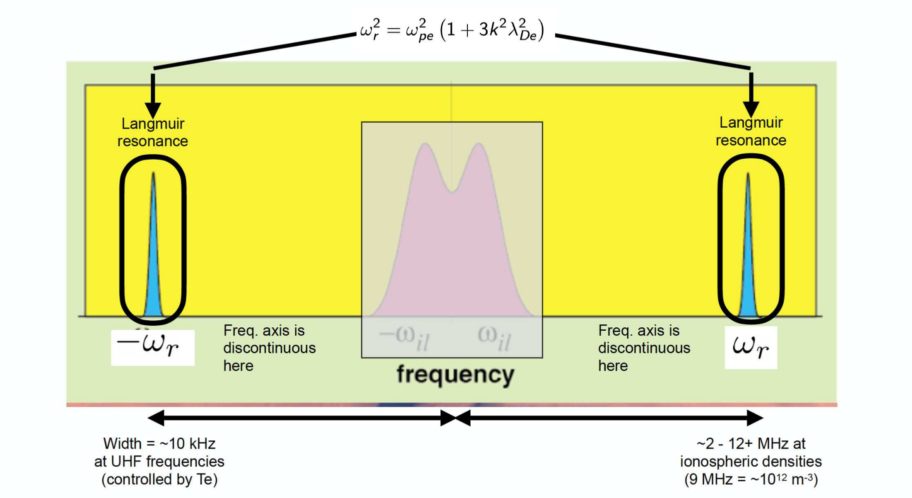
$$\left\langle \left| n_{ti}\left(\mathrm{k},\omega\right)\right|^2 \right\rangle \qquad \frac{|\chi_e|^2}{|1+\chi_i+\chi_e|^2}$$



IS Spectrum: Ion Line



IS Spectrum: Plasma Line



Dispersion Relation Summary

- IS theory is closely related to theory of electrostatic plasma waves
- Peaks in the spectrum when $\omega = k_b v_\phi$
- Ion line peaks are ion acoustic waves
- Plasma line peaks are Langmuir waves