Basic Radar 3.5: Ambiguity Functions

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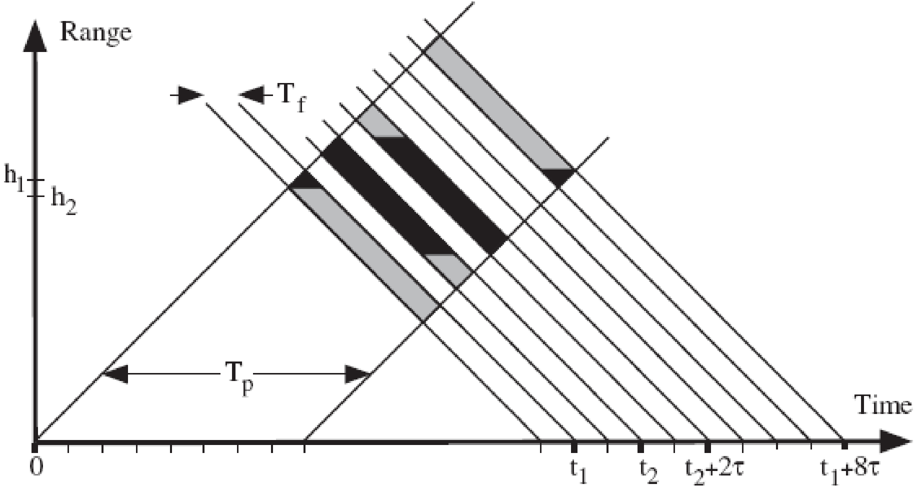
How are ACFs and LPAs Related?

- Given $N_e(r)$, $T_e(r)$, $T_i(r)$, and $u_{\mathrm{los}}(r)$ as functions of range, you can theoretically compute a 2D ACF $A(r,\tau)$ as a continuous function of range and lag time.
- The LPA is a 2-index array where the i approximately represents range and ℓ approximately represents lag time.

$$L_\ell^i \equiv \left\langle V^*_{i-\lfloor\frac{\ell}{2}\rfloor} V_{i+\lfloor\frac{\ell}{2}\rfloor+(\ell \bmod 2)} \right\rangle$$

Ambiguity function theory defines the mathematical relationship between these two quantities.

LPA Mixes Range and Time Information



Each lagged product contains information smeared in range over the dark overlap regions.

Amplitude Ambiguity Function

After transmitting a pulse envelope $s(t)$, the scattered signal is

$$x(t) = \int d^3\mathrm{r}\, e^{j\mathrm{k}\cdot r} s\left(t - \frac{2r}{c}\right) \Delta N_e \left(\mathrm{r}, t - \frac{r}{c}\right)$$

The receiver records a filtered and sampled version of the scattered signal

$$\begin{aligned} y\left(t_s\right) &= \int dt\, x(t) h^*\left(t_s - t\right) \\ &= \int dt d^3\mathrm{r}\, e^{j\mathrm{k}\cdot\mathrm{r}} s\left(t - \frac{2r}{c}\right) \Delta N_e \left(\mathrm{r}, t - \frac{r}{c}\right) h^*\left(t_s - t\right) \end{aligned}$$

Define the **amplitude ambiguity function**

$$W_{t_s} \equiv s\left(t - \frac{2r}{c}\right) h^*\left(t_s - t\right)$$

$$y\left(t_s\right) = \int dt d^3\mathrm{r}\, e^{j\mathrm{k}\cdot\mathrm{r}} W_{t_s}(t, r) \Delta N_e \left(\mathrm{r}, t - \frac{r}{c}\right)$$

Range-Lag Ambiguity Function

When we form ACFs, we take products of samples and average:

$$\langle y(t_{s2})\, y^*(t_{s1})\rangle = \int dt_1 dt_2 d^3\mathrm{r}_1 d^3\mathrm{r}_2 e^{j\mathrm{k}\cdot(\mathrm{r}_2-\mathrm{r}_1)} \left\langle \Delta N_e\left(\mathrm{r}_2, t_2 - \frac{r_2}{c}\right) \Delta N_e^*\left(\mathrm{r}_1, t_1 - \frac{r_1}{c}\right)\right\rangle W_{ts2}(t_2, r_2)\, W_{ts1}^*(t_1, r_1)$$

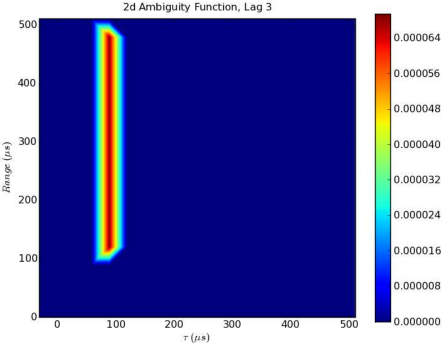
Change variables $t_1 = t \quad t_2 = t + \tau \quad \mathrm{r}_1 = \mathrm{r} \quad \mathrm{r}_2 = \mathrm{r} + \mathrm{r}'$
Perform r' integral and take expected value

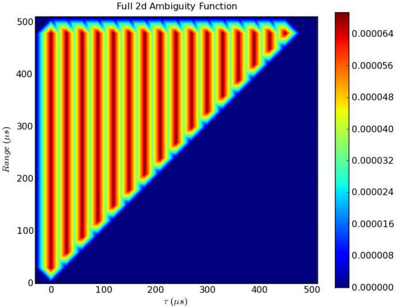
$$\langle y(t_{s2})\, y^*(t_{s1})\rangle = \int d\tau d^3\mathrm{r}\, A(r,\tau) \underbrace{\int dt\, W_{ts2}(t+\tau, r)\, W_{ts1}^*(t, r)}_{W_{t_{s1},t_{s2}}(\tau,r)}$$

The measured lag-product is the ISR ACF we want $A(r,\tau)$ blurred the the **range-lag ambiguity function** $W_{t_{s1},t_{s2}}(\tau, r)$

2-D Range-lag Ambiguity Function of Long Pulse

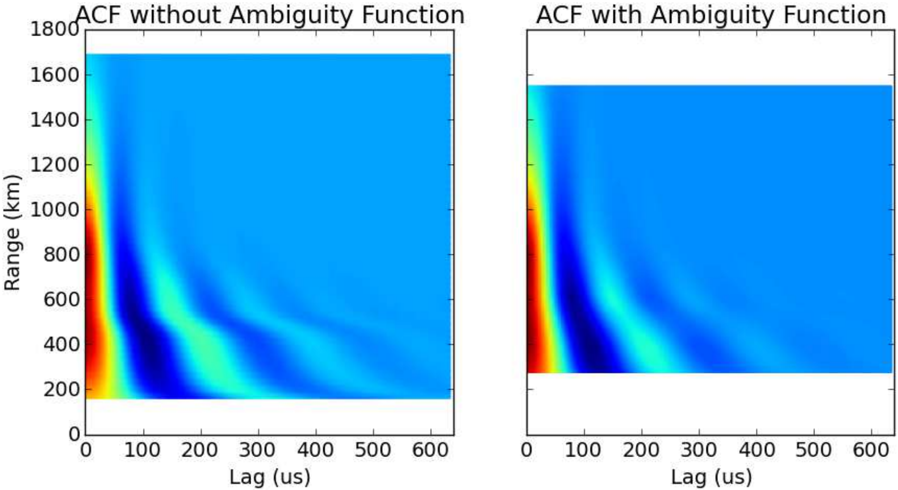
Ambiguity function with a boxcar filter. 480 μs long pulse, 30 μs sampling.



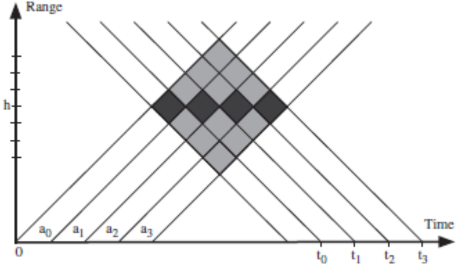


Theoretical Long Pulse Examples

A particular exaggerated example using 1.5 ms long pulses and a profile with a sharp T_e gradient at 500 km.



Coded Pulse Experiments

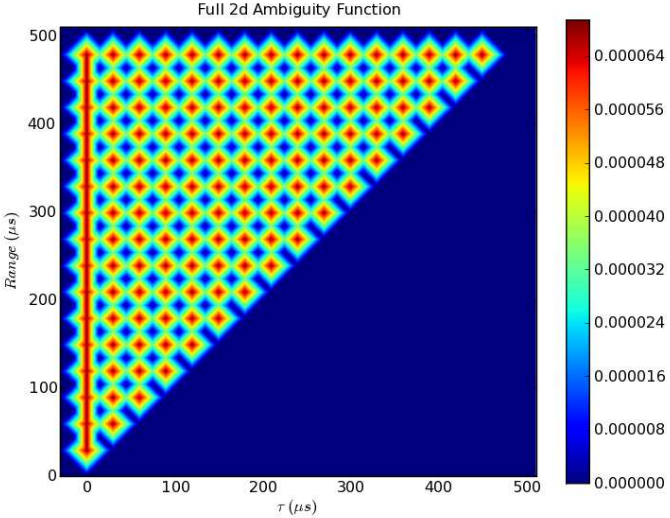


$$a_0 a_1 v_0 v_1^* = a_0\left(a_0 s_h^t + a_1 s_{h-1}^{t+\frac{1}{2}} + a_2 s_{h-2}^{t+1} + a_3 s_{h-3}^{t+\frac{3}{2}}\right)\times$$
$$a_1\left(a_0 s_{h+1}^{t+\frac{1}{2}} + a_1 s_h^{t+1} + a_2 s_{h-1}^{t+\frac{3}{2}} + a_3 s_{h-2}^{t+2}\right)^*$$

$$E\left\{a_0 a_1 v_0 v_1^*\right\} = E\left\{s_h^t s_h^{*t+1}\right\} + a_0 a_2 E\left\{s_{h-1}^{t+\frac{1}{2}} s_{h-1}^{*t+\frac{3}{2}}\right\}$$
$$+ a_0 a_1 a_2 a_3 E\left\{s_{h-2}^{t+1} s_{h-2}^{*t+2}\right\}$$

Range-lag Ambiguity Function of Alternating Codes

Ambiguity function for a boxcar filter. 480 μs (16-baud, 30 μs baud, 32 pulse).



Ambiguity Functions Summary

- Ambiguity functions mathematically describe the relationship between the plasma ACF (a function of N_e, T_e, T_i, u_{los}) and the lagged products we actually estimate.
- Ambiguity functions are only a function of the transmit waveform (s) and the receiver impulse response (h).
- Ambiguity functions act like blurring functions that smear information out in range.
- In long pulse experiments the extent of blurring is related to the pulse length.
- In coded pulse experiments the extent of blurring is related to the baud length.