Basic Radar 3.3: Introduction to Stochastic Processes

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Stochastic Processes: Definitions and Terminology

- Stochastic Process (aka Random Process):
  $V(t)$ where value at every time is a random variable
- Gaussian Stochastic Process:
  - PDF of each $V(t)$ is a Gaussian distribution (aka normal distribution)
  - Joint PDF of any subset of samples of $V(t)$ is a jointly Gaussian distribution (aka Multivariate Normal Distribution)
- Moments of a Stochastic Process:
  - Mean: $\bar{V}(t) = E\left\{V(t)\right\}$
  - Autocorrelation: $R_V\left(t, t-\tau\right) = E\left\{V(t)V^*(t-\tau)\right\}$
  - Autocovariance:
    $C_V\left(t, t-\tau\right) = E\left\{\left[V(t) - \bar{V}(t)\right]\left[V^*(t-\tau) - \bar{V}^*(t-\tau)\right]\right\} = R\left(t, t-\tau\right) - \bar{V}(t)\bar{V}^*(t-\tau)$
- (Wide Sense) Stationary Stochastic Process
  - $\bar{V}(t) = \bar{V}$ is independent of t
  - $R(t, t-\tau) = R(\tau)$ is independent of t
- ISR signals are Gaussian, zero mean, and stationary as long as the ionospheric state parameters are constant.

Power Spectra of Deterministic Signals

Given a signal $f(t)$ and its fourier transform
$F(\omega) = \mathcal{F}\left\{f(t)\right\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$, the power spectrum is:

$$\begin{aligned} S_F(\omega) = \left|F(\omega)\right|^2 &= F^*(\omega)F(\omega) \\ &= \mathcal{F}\left\{f(-t') * f(t')\right\} \\ &= \mathcal{F}\left\{\int_{-\infty}^{\infty} f(t')f(t'-t)\,dt'\right\} \end{aligned}$$

When you filter a signal:

$$\begin{aligned} g(t) &= h(t) * f(t) \\ G(\omega) &= H(\omega)F(\omega) \\ S_G(\omega) &= \left|H(\omega)\right|^2 S_F(\omega) \end{aligned}$$

Power Spectra of Stochastic Signals

Fourier transforms of stationary random processes do not exist.
Fourier transforms of ACFs will exist, and are the power spectra:

$$S_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau) e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} E\{V(t)V^*(t-\tau)\} e^{-j\omega\tau}\, d\tau$$

Properties:

- $S(\omega)$ is real and $S(\omega) \geq 0$
- Short correlation times $\leftrightarrow$ wide bandwidth and vice versa
- $\int_{-\infty}^{\infty} S_V(\omega)\, d\omega = R(0) = E\{|V|^2\}$ (total power)
- If U = h * V, $S_U(\omega) = |H(\omega)|^2\, S_V(\omega)$

Intuitive interpretation: $\int_{\omega_1}^{\omega_2} S_V(\omega)\, d\omega$ is the power in the frequency band from ω_1 to ω_2.

Example: Running Average of White Noise

Continuous white noise:

$$E\left\{W(t)\right\} = 0 \qquad S_W(\omega) = S_0 \qquad R_W(\tau) = S_0\delta(\tau)$$

Running average of white noise:

$$V(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} W(t')\,dt'$$

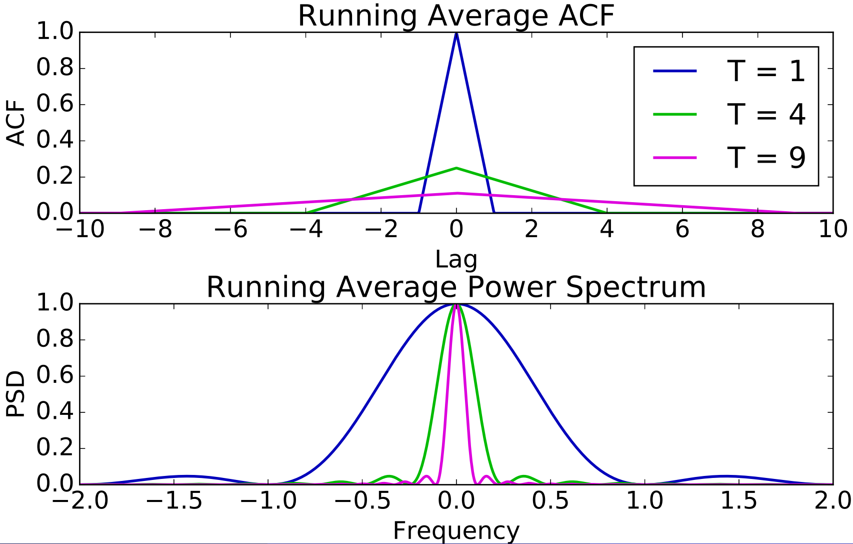
$$R_V\left(\tau\right) = E\left\{\frac{1}{T}\int_{t-T/2}^{t+T/2} W(t')\,dt'\frac{1}{T}\int_{t+\tau-T/2}^{t+\tau+T/2} W(t'')\,dt''\right\}$$

$$= \frac{1}{T^2}\int_{t-T/2}^{t+T/2}\int_{t+\tau-T/2}^{t+\tau+T/2} S_0\delta\left(t'-t''\right)\,dt''dt'$$

$$= \begin{cases} S_0\frac{T-|\tau|}{T^2} & |\tau| < T \\ 0 & |\tau| \geq T \end{cases} \quad\Rightarrow\quad S_V(\omega) = S_0\left(\frac{\sin\left(\omega T/2\right)}{\omega T/2}\right)^2$$

Correlation Time and Bandwidth

Short correlation times → wide-bandwidth and vice versa



Examples of Discrete Stochastic Process

Gaussian white noise W_n:

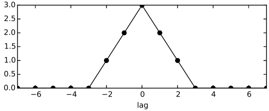
$$\bar{W} = 0 \qquad R_\ell = E\left\{W_n W^*_{n-\ell}\right\} = \begin{cases} \sigma_0^2 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

3-point running sum of Gaussian white noise

$$V_n = W_n + W_{n-1} + W_{n-2}$$

$$R_\ell = E\left\{[W_n + W_{n-1} + W_{n-2}]\,[W_{n-\ell} + W_{n-\ell-1} + W_{n-\ell-2}]^*\right\}$$

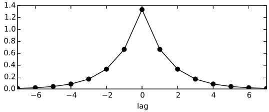
$$= \begin{cases} (3 - |\ell|)\,\sigma_0^2 & |\ell| < 3 \\ 0 & |\ell| \geq 3 \end{cases}$$



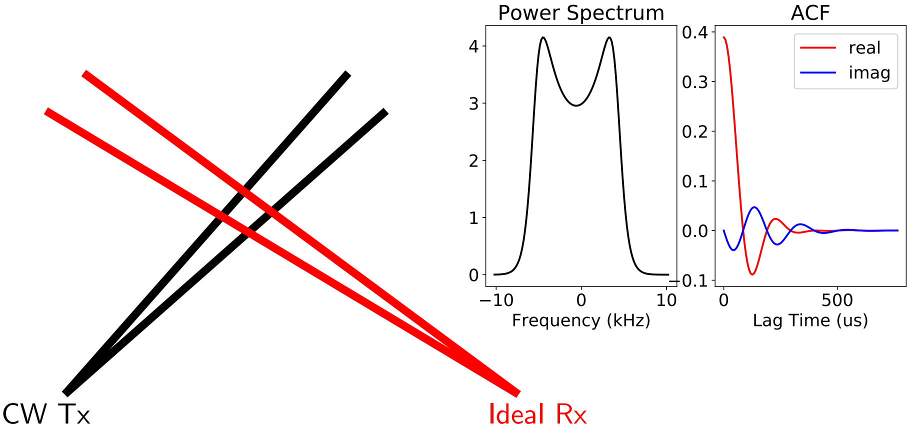
Autoregressive model

$$Y_n = \alpha Y_{n-1} + W_n \qquad Y_n = W_n + \alpha W_{n-1} + \alpha^2 W_{n-2} + \alpha^3 W_{n-3} + \cdots$$

$$R_\ell = \frac{\alpha^{|\ell|}}{1 - \alpha^2}\sigma_0^2$$



A Hypothetical CW Bistatic ISR Experiment



ISR theory gives the PSD and ACF of the received voltages as a function of N_e, T_e, T_i, and u_{los} in the overlap volume.

Stochastic Process Summary

- Generalize random variables and vectors of random variables to "functions" of time that are random variables.
- ACF of a stochastic process plays a similar role to the covariance matrix of a vector of random variables.
- Power spectrum and ACF are a Fourier transform pair.