# Radar Signal Processing: Part 4

## Essential Frequency Domain Concepts for ISR

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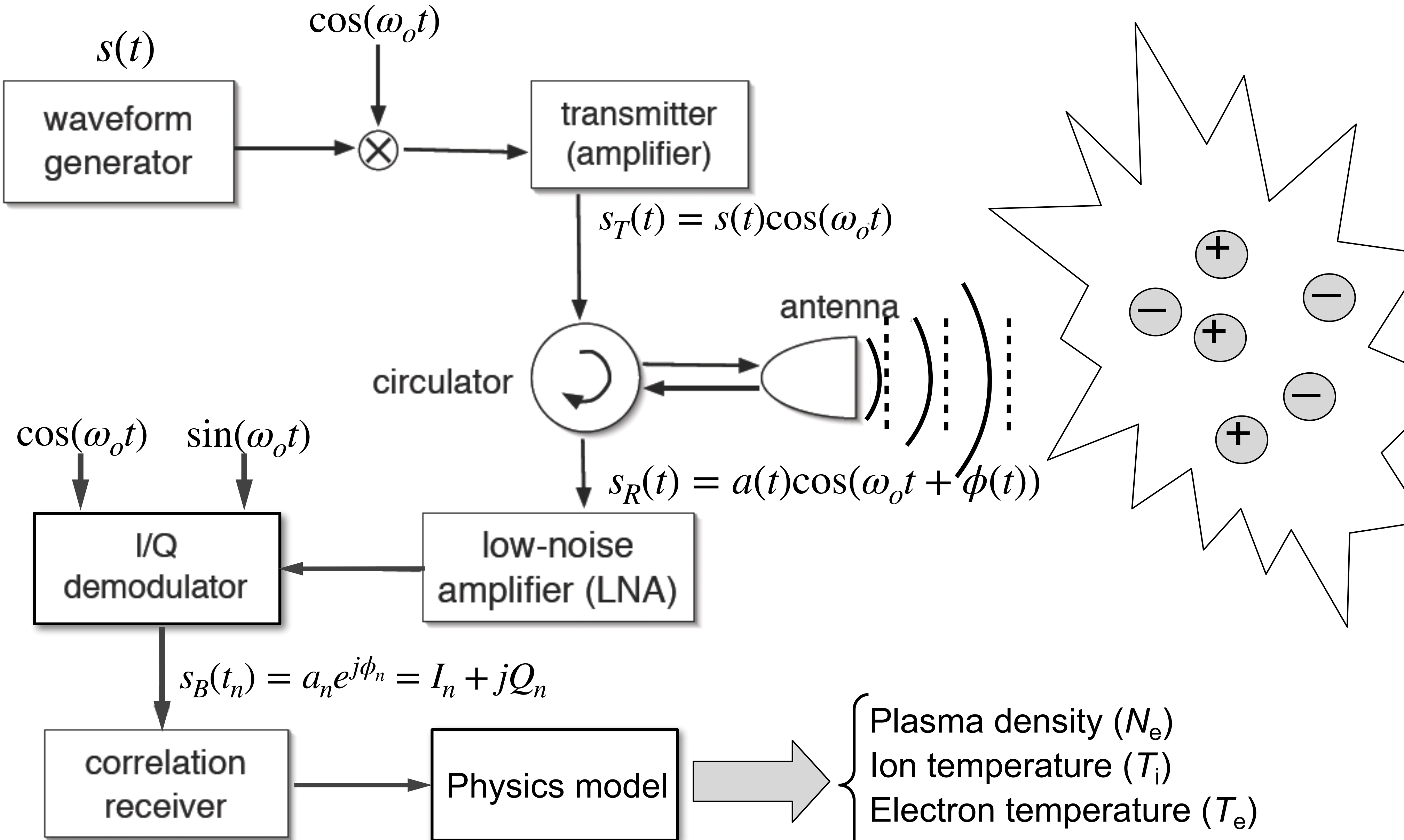
# Components of a Pulsed Doppler Radar




- Plasma density ($N_e$)
- Ion temperature ($T_i$)
- Electron temperature ($T_e$)
- Bulk velocity ($V_i$)

# Essential mathematical operations

**Fourier Transform:** Expresses a function as a weighted sum of complex exponentials.

analysis equation: $$F(\omega) = \mathcal{F}\left[f(t)\right] = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

synthesis equation: $$f(t) = \mathcal{F}^{-1}\left[F(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}d\omega$$

$$f(t) \quad \Longleftrightarrow \quad F(\omega)$$

**Duality:** Comparison of $\mathcal{F}$ and $\mathcal{F}^{-1}$ we obtain

$$F(t) \quad \Longleftrightarrow \quad 2\pi f(-\omega)$$

**Convolution:** Expresses the action of a linear, time-invariant system on a function.

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$$

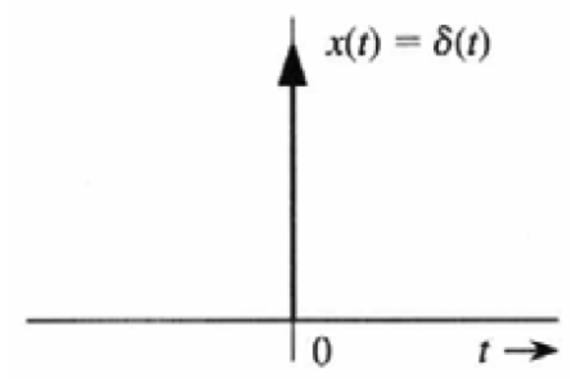
$$f(t) * g(t) \quad \Longleftrightarrow \quad F(\omega)G(\omega)$$

$$f(t)g(t) \quad \Longleftrightarrow \quad F(\omega) * G(\omega)$$

# Dirac Delta Function $\delta(x)$

A generalized function, or distribution, with the properties

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \qquad \int_{-\infty}^{+\infty} \delta(x)dx = 1$$




**Sampling property:** From the above it follows that

$$f(t_o) = \int_{-\infty}^{+\infty} f(t)\delta(\underbrace{t - t_o}_{})dt$$

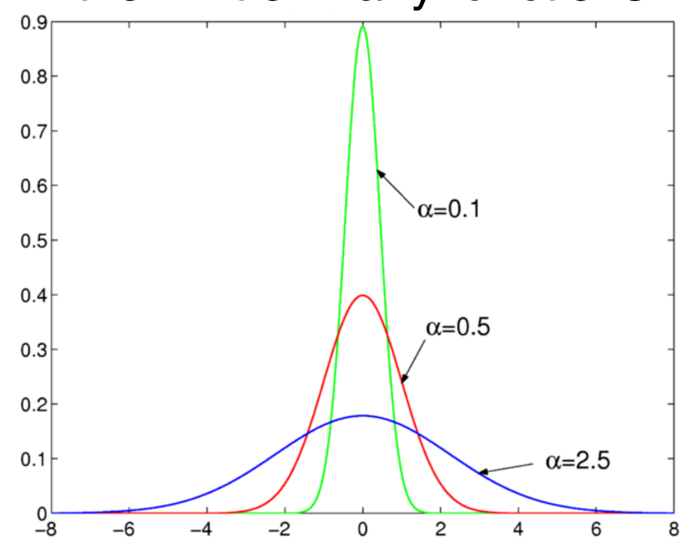
argument is zero at $t = t_0$

**Shift property:** Convolution of a function $F(x)$ with $\delta(x - x_o)$ shifts the entire function by $x_o$. We will use this property to understand mixing. Specifically:

$$F(\omega) * \delta(\omega - \omega_0) = \int_{-\infty}^{+\infty} F(\Omega)\delta(\underbrace{\omega - \omega_0}_{} - \Omega)d\Omega$$

$$= F(\omega - \omega_0)$$

$\delta(t)$ may be expressed as the limit of many functions

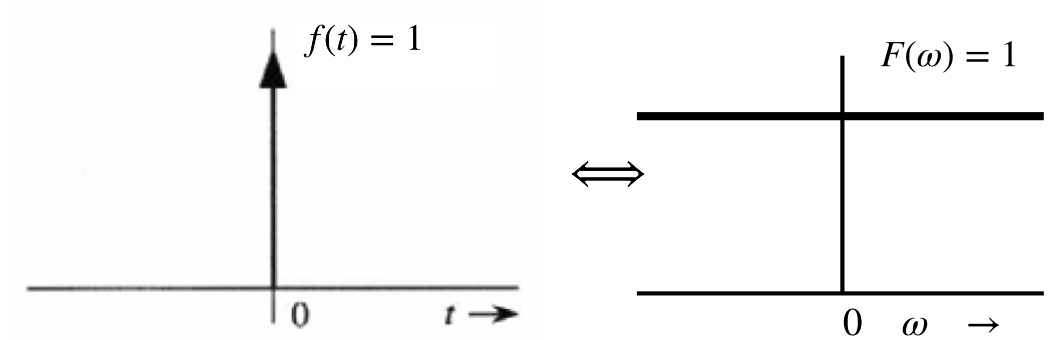



$$\delta(t) = \lim_{\alpha \to 0} \frac{1}{\sqrt{4\pi\alpha}} e^{-t^2/(4\alpha)}$$

# Fourier analysis of harmonic functions

$$\mathscr{F}\left[\delta(t)\right] = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt \; = e^{-j0} dt$$

$$\boxed{\mathscr{F}\left[\delta(t)\right] = 1}$$




$$\mathscr{F}\left[\delta(t - t_o)\right] = \int_{-\infty}^{+\infty} \delta(t - t_o) e^{-j\omega t} dt$$

$$\boxed{\mathscr{F}\left[\delta(t - t_o)\right] = e^{-j\omega t_o}}$$

From duality property we can also write,

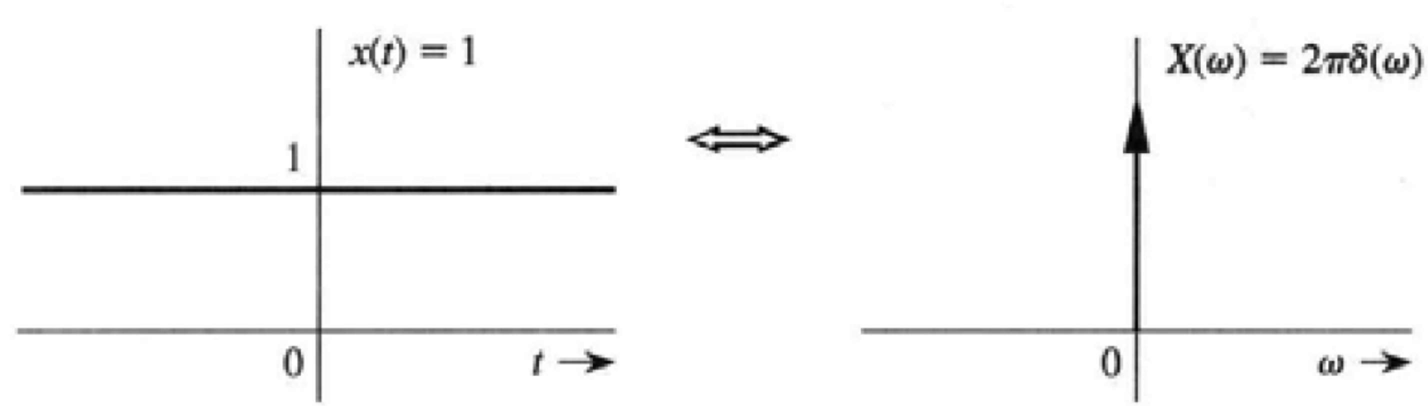
$$\boxed{\mathscr{F}[1] = 2\pi\delta(\omega)} \qquad \boxed{\mathscr{F}\left[e^{j\omega_o t}\right] = 2\pi\delta(\omega - \omega_o)}$$

$$f(t) = \cos(\omega_o t) = \frac{1}{2}\left[e^{j\omega_o t} + e^{-j\omega_o t}\right]$$

$$F(\omega) = \int_{-\infty}^{+\infty} \frac{1}{2}\left[e^{j\omega_o t} + e^{-j\omega_o t}\right] e^{-j\omega t} dt \; = \frac{1}{2}\left[\int_{-\infty}^{+\infty} e^{j\omega_o t} e^{-j\omega t} dt + \int_{-\infty}^{+\infty} e^{-j\omega_o t} e^{-j\omega t} dt\right]$$

$$\boxed{\mathscr{F}\left[\cos(\omega_o t)\right] = \pi\left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)\right]}$$

# Fourier analysis of harmonic functions

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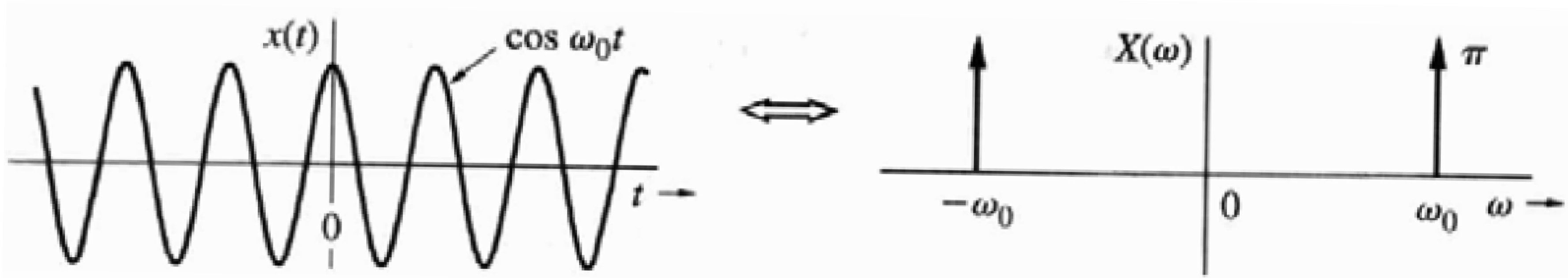
$$\cos \omega_0 t \iff \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

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$$\sin \omega_0 t \iff j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

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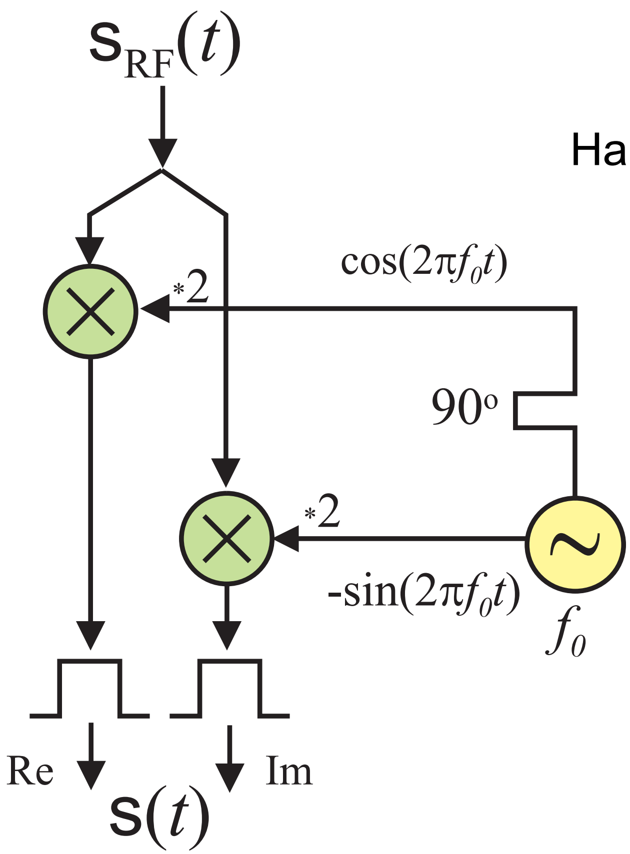
$$e^{j\omega_o t} = \cos(\omega_o t) + j\sin(\omega_o t) \iff 2\pi\delta(\omega - \omega_0)$$

# Summary of tools for I/Q demodulation

Multiplication-convolution: $f(t)g(t) \Longleftrightarrow F(\omega) * G(\omega)$

Frequency shift property: $F(\omega) * \delta(\omega - \omega_0) = F(\omega - \omega_0)$




Harmonic functions:

$$\cos(\omega_o t) \Longleftrightarrow \pi\left[\delta(\omega + \omega_o) + \delta(\omega - \omega_o)\right]$$

$$\sin(\omega_o t) \Longleftrightarrow j\pi\left[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)\right]$$

$$e^{j\omega_o t} \Longleftrightarrow 2\pi\delta(\omega - \omega_o)$$

# I/Q Demodulation: Frequency Domain

Transmitted signal:

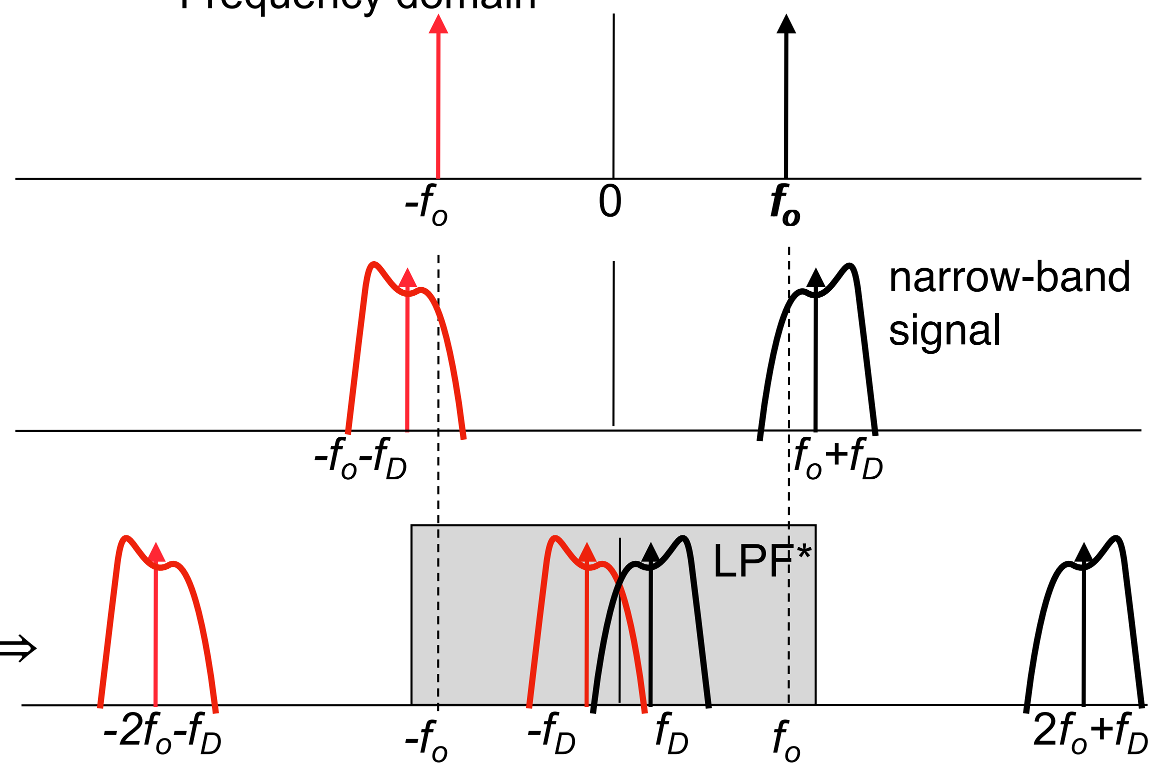
$$\cos(2\pi f_o t) \iff$$

Reflected signal from moving target

$$\cos(2\pi(f_o + f_D)t) \iff$$

Mixed (multiplied) with oscillator $\cos(2\pi f_o t)$

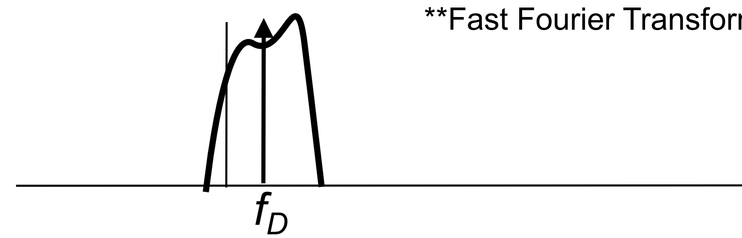
$$\frac{1}{2}\cos\left[2\pi(2f_o + f_D)t\right] + \frac{1}{2}\cos[2\pi f_D t] \iff$$




*Low Pass Filter
**Fast Fourier Transform

To resole both positive and negative Doppler shifts, we need:

$$e^{j2\pi f_D t} = \cos(2\pi f_D t) + j\sin(2\pi f_D t)$$




We thus need to mix with a second oscillator at same frequency but 90° out of phase (Lecture 3). For a cosine reference, the quadrature function is sine. The two components are called “in phase” (*I*) and “quadrature” (*Q*). Together *I* and *Q* represent discrete samples of the baseband analytic signal,

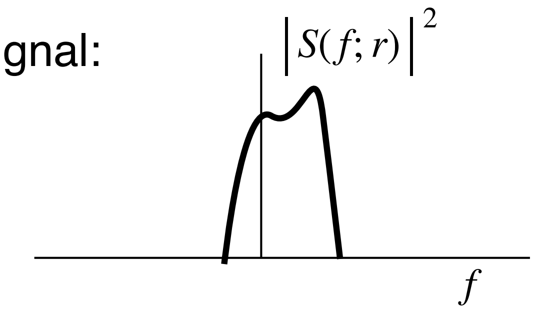
$$s_B(t) = Ae^{2\pi f_D t} = I(t) + jQ(t) \quad \overset{\text{FFT**}}{\iff} \quad A\delta(f - f_D)$$

(for a single scatterer)

# Correlation and the ISR Spectrum

How do we compute the power spectrum from our complex voltages ?

One approach is to compute Fourier transform of the range-resolved signal:




$$s(r,t) = I(r,t) + Q(r,t) \quad \Longleftrightarrow \quad S(r,f)$$

from which the power spectrum may be represent as $\left|S(r,f)\right|^2$

Based on the stochastic nature of the target, and the way ISR samples the echos, we will take a different approach. We first compute the auto-correlation function (ACF),

$$R_s(r,\tau) = \frac{\left\langle s(r,t)\overline{s(r,t+\tau)}\right\rangle}{\left\langle \left|s(r,t)\right|^2\right\rangle}$$

where the angle brackets denote the ensemble average, or the expected value. The power spectral density is given by the Fourier transform of the $R_s$

$$R_s(r,\tau) \Longleftrightarrow \left|S(r,f)\right|^2 \qquad \text{(Wiener-Khinchin theorem)}$$

The discrete representation of $R_s(r,\tau)$ is constructed through appropriate scaling and multiplication of the complex voltage samples $s(r_k, t_n)$.

In the next lecture we will begin to explore methods for constructing the ACF.