Dispersion Relations And the IS Spectral Shape

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Topics covered:

- Dispersion relation concept and examples
- Important dispersion examples for the ionospheric plasma
- Connections to incoherent scatter spectral properties

Dispersion relation: the concept

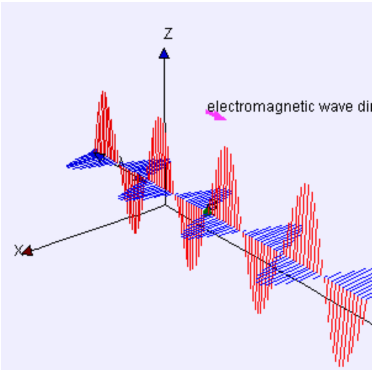
Key concept for wave behavior within a propagation medium.
Its functional form encodes physical properties of a medium and their parameter dependence.

For wave behavior, describes the relationship between SPATIAL frequency (wavelength) and TEMPORAL frequency in the medium.

Some media relate wavelength to frequency **linearly (dispersionless),**
but waves in most media have **nonlinear** relation between wavelength and frequency.

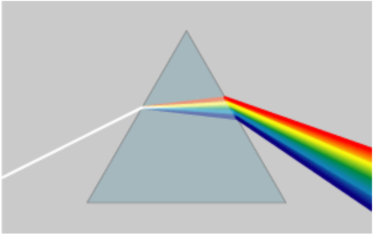
Dispersionless example:
EM radiation propagation through free space
(wavelength / velocity = c)

$$\omega(k) = c\ k$$



http://weelookang.blogspot.com/2011/10/ejs-open-source-propagation-of.html

Nonlinear dispersion example:
splitting of light through a prism
(effective speed of light depends on wavelength
due to material properties of glass)



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Dispersion relation: Phase and Group Velocity

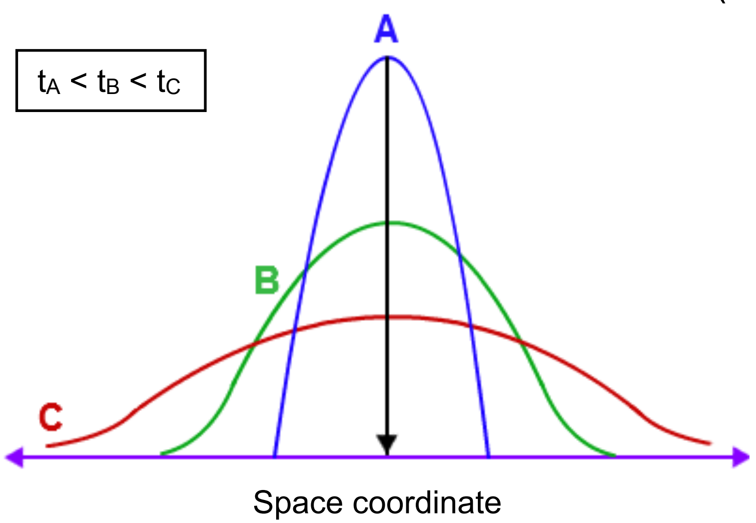
Knowing the dependence of wavelength as a function of frequency allows us to define:

$$V_{phase} = \omega / k \qquad\qquad V_{group} = \delta\omega / \delta k$$

Speed of a single frequency sinusoidal wave

Speed of a **collection** of sinusoidal waves
Speed of information travel / modulation envelope in comm. theory

(NB: group shape will change over time if dispersion happens)

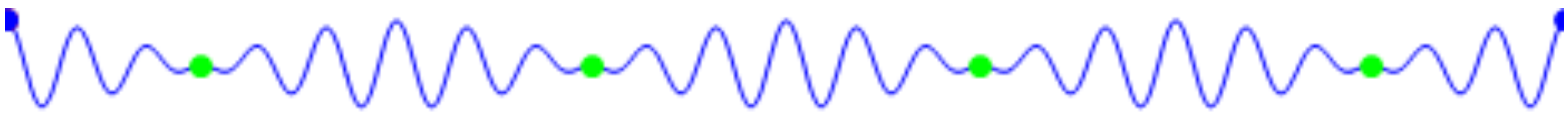


Example of pulse spreading spatially from time A to B to C due to dispersion.

http://www.mathcaptain.com/statistics/dispersion-statistics.html

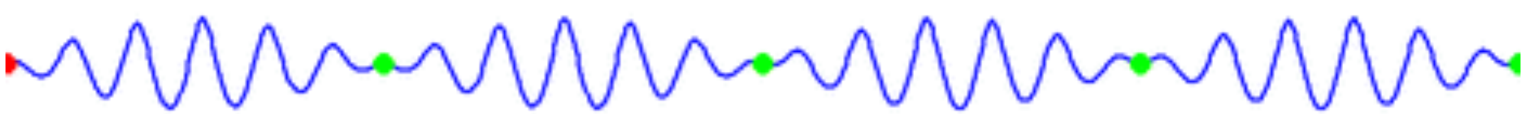
Dispersion relation: the concept

Dispersion free travel in a transverse wave with 2 frequencies:
Note that phase (red) velocity = group (green) velocity



Unit sphere / CC-BY-SA-3.0

Nonlinear dispersion in a transverse wave with 2 frequencies:
Note that phase (red) velocity is **faster** here than group (green) velocity
(in other cases, phase velocity might be slower than group velocity)



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Plasma dispersion relations: Collective effects

$$\epsilon(\omega, \vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Dielectric constant of the medium

Phase velocity! (Doppler spectrum is important)

The physics of the medium is described by the dielectric constant (related to plasma conductivities)

in free space:
H = B D = E

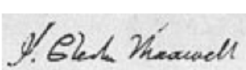
Gauss' Law (electric field around charges) $\nabla \cdot \mathbf{D} = \rho_f$

Gauss' Law for magnetism (no magnetic monopoles) $\nabla \cdot \mathbf{B} = 0$

Faraday's Law (electric field around a changing magnetic field) $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

Ampere's Law (magnetic field circulation around electric charges) $\nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}$

Maxwell's correction (displacement current)



(circles = places where dielectric constant shows up in Gauss, Ampere)

Important thermal plasma dispersion relations

$$\epsilon(\omega, \vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Dielectric constant of the medium

Insert plasma dispersion relation here

1) *Ion-acoustic fluctuations* [sound waves in plasma]

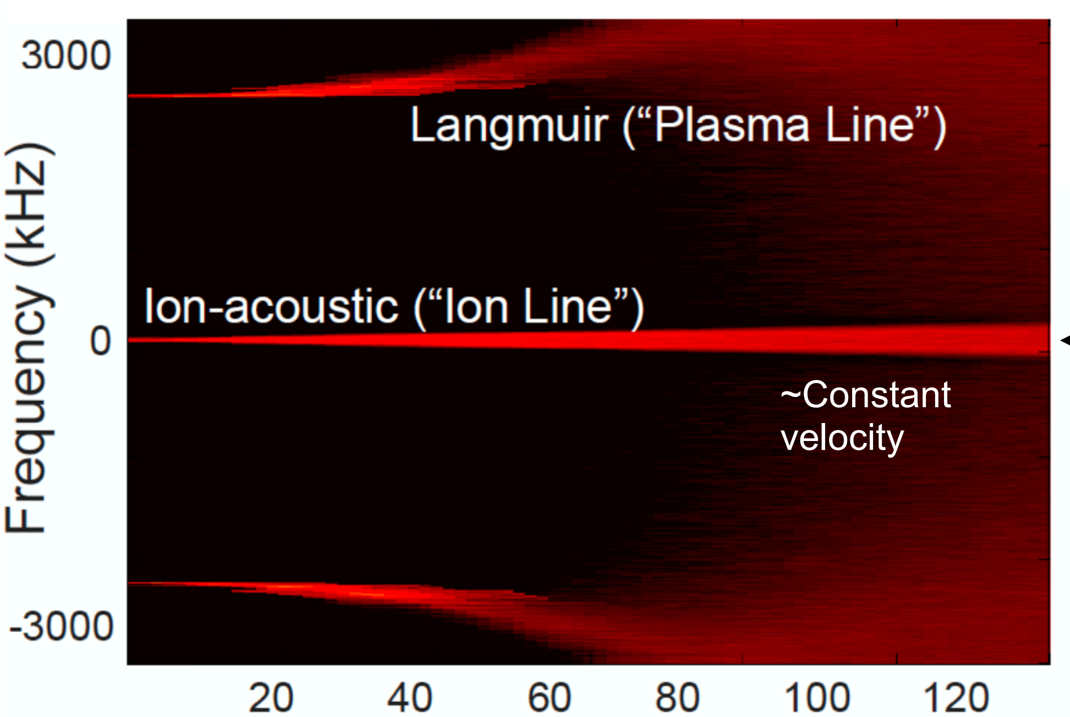
$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = V_s$$

NB: ordinary acoustic waves: adiabatic compression / decompression of fluid particles. Stiff medium transmits forces.

Ion-acoustic fluctuations:
restoring force = electrostatic
(so applicable in F region; near collisionless)

Important thermal plasma dispersion relations

$$\left\langle \left| n_e\left(\mathrm{k},\omega\right)\right|^2 \right\rangle$$



Wavelength

Diaz et al [2008]

$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = V_s$$

Important thermal plasma dispersion relations

$$\epsilon(\omega, \vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Insert plasma dispersion relation here

2) ***Langmuir oscillations*** **(Plasma oscillations):**

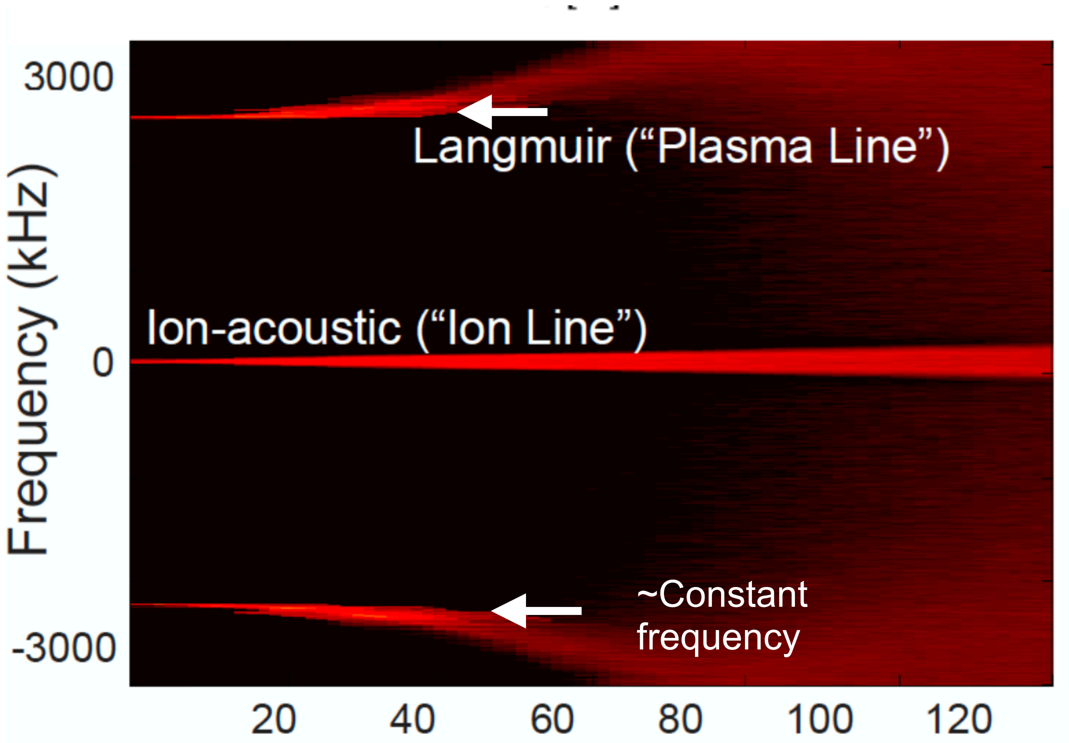
$$\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_{th}^2$$

$$v_{th}^2 = 2k_B T_e/m_e$$

Akin to Brunt-Våisålå oscillations
in fluid (parcel in presence of density gradient) -
here, electrostatic field is restoring force, and
electron pressure gradient transmits information

Important thermal plasma dispersion relations

$$\left\langle \left| n_e\left(\mathrm{k},\omega\right)\right|^2 \right\rangle$$



Wavelength

Diaz et al [2008]

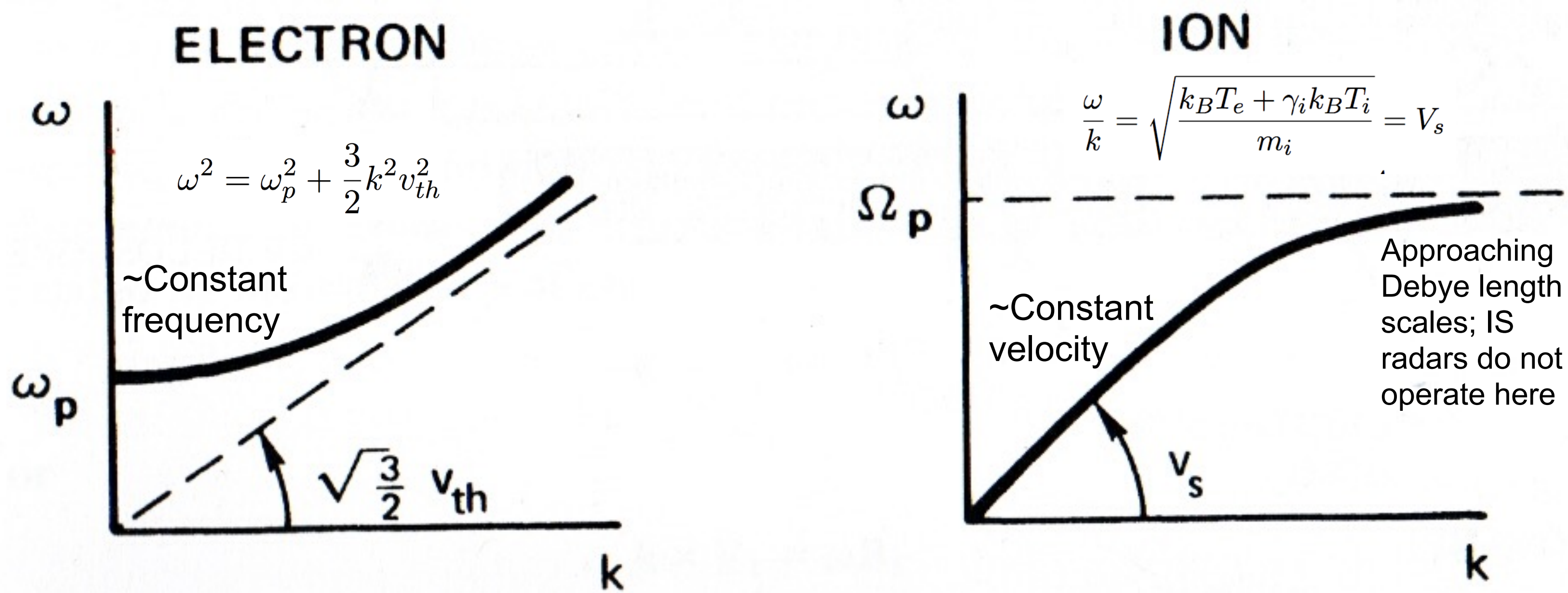
$$\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_{th}^2$$

$$v_{th}^2 = 2k_B T_e / m_e$$

Important dispersion relations for incoherent scatter

Langmuir waves:
Mostly constant **frequency**
Electron density and
electron temperature

Ion-acoustic waves:
Mostly constant **velocity**
Ion mass, ion and electron
temperature

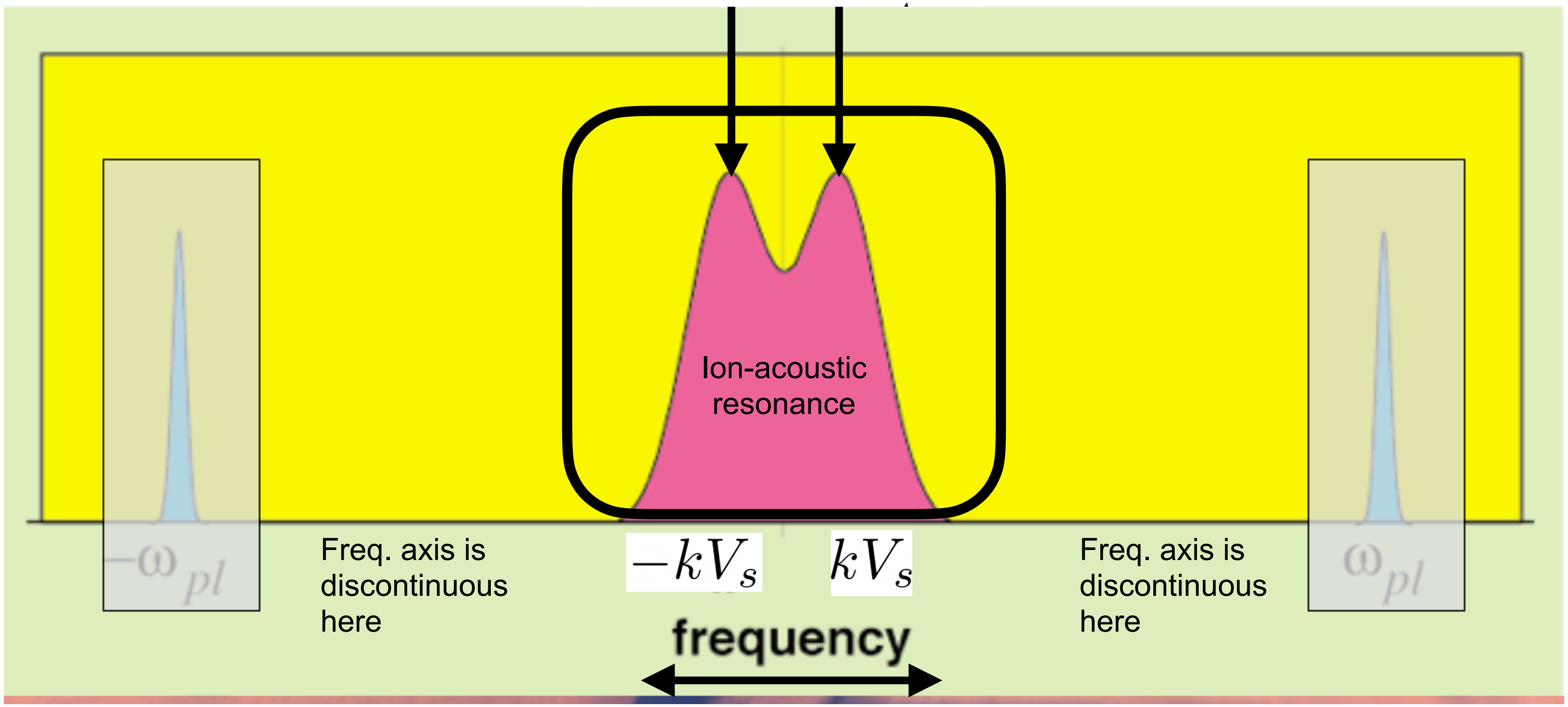


Comparison of the dispersion curves for electron plasma waves and ion acoustic waves.

(Chen, Intro to Plasma Physics)

Ion-acoustic resonance: IS spectral shape

$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = V_s$$

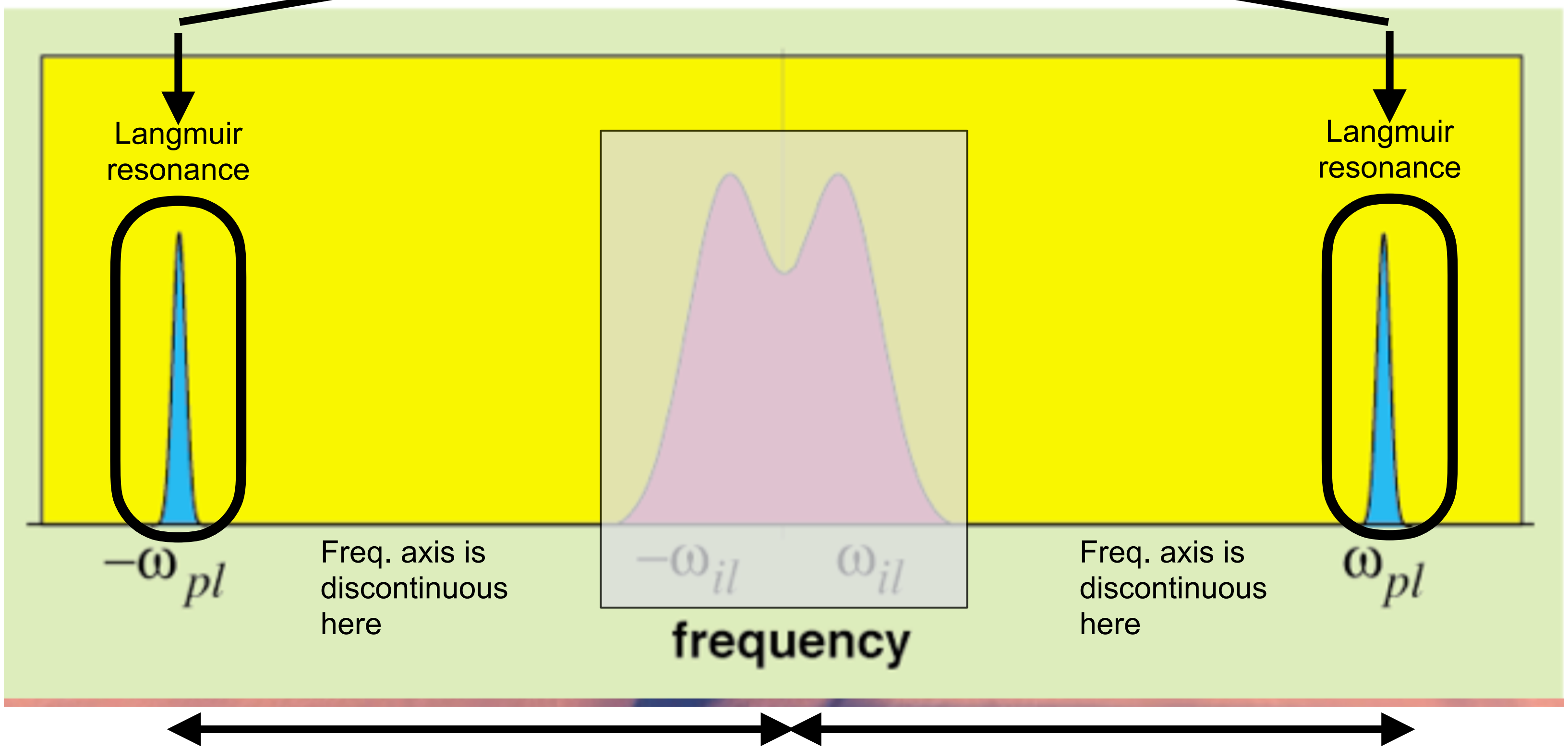


~10 kHz @ UHF frequencies

Why are the peaks finite width and not a delta function? - **Landau damping** (cf. other lecture)

Langmuir resonance: IS spectral shape

$$\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_{th}^2$$



Width = ~10 kHz at UHF frequencies (controlled by Te)

~2 - 12+ MHz at ionospheric densities (9 MHz = ~10^{12} m^{-3})

Dispersion Relations And the IS Spectral Shape

Summary

- Plasmas have important nonlinear dispersion relations
- These govern the plasma resonant response to thermal energy input
- Thermal ionospheric plasma parameters set key aspects of the observed incoherent scatter spectral shape
- Remote sensing of the ionosphere can be done!