ISR Theory: Thomson Scattering

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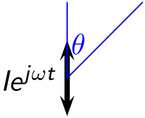
Hertzian Dipole Antenna

Consider an infinitesimal dipole antenna of length $d\ell$ carrying current a sinusoidal current I



$$\mathrm{J} = Id\ell\,\delta\left(\mathrm{x}\right)\hat{z}e^{j\omega t}$$

Far Field Solution ($\eta_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\ \Omega$)

$$\mathrm{E}_{ff} = \frac{jk_0\eta_0 Id\ell}{4\pi r}\sin\theta e^{j\omega t - jk_0 r}\hat{\theta}$$

$$\mathrm{B}_{ff} = \frac{jk_0\mu_0 Id\ell}{4\pi r}\sin\theta e^{j\omega t - jk_0 r}\hat{\phi}$$

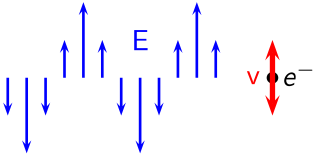
Far Field Radiated Power

$$S = \frac{1}{2\mu_0}\Re\left\{\mathrm{E}\times\mathrm{B}^*\right\} = \frac{1}{2\eta_0}\left|E\right|^2 = \frac{\eta_0}{2}\left(\frac{k_0 Id\ell}{4\pi r}\right)^2\sin^2\theta$$

Thomson Scatter from One Electron

Incident wave:

$$\mathrm{E} = \hat{z} E_0 e^{j\omega t - j\mathrm{k}_0 \cdot \mathrm{r}}$$



Motion of the electron:

$$j\omega m_e \mathrm{v} = -e\mathrm{E} \rightarrow \mathrm{v} = \frac{je}{\omega m_e} E_0 \hat{z}$$

Effective Hertzian Dipole with $Id\ell \rightarrow e\mathrm{v}$ (also note $\omega / k_0 = c$)

$$\mathrm{E}_{scat} = \frac{-\eta_0 e^2}{4\pi r m_e c} E_0 \sin\theta e^{j\omega t - j\mathrm{k}_0 \cdot \mathrm{r}} \hat{\theta} = -\frac{r_e}{r} E_0 e^{j\omega t - j\mathrm{k}_0 r} \hat{\theta}$$

Where the classical electron radius is

$$r_e = \frac{\eta_0 e^2}{4\pi m_e c} = \frac{e^2}{4\pi \epsilon_0 m_e c^2} \approx 2.818 \times 10^{-15} \ \mathrm{m}$$

Thomson Scatter Cross Section

Total Cross Section:

$$\sigma_t \equiv \frac{P_{\text{tot}}}{\frac{1}{2\eta_0}\left|E_0\right|^2} = \frac{8\pi}{3} r_e^2 \qquad \text{Where } P_{\text{tot}} \equiv \int_0^{2\pi}\int_0^{\pi} S_{scat}\, r^2 \sin\theta \, d\theta d\phi$$

Radar Cross Section:

$$\sigma = \sigma_t D_s$$

Directivity of scattering in the direction towards the radar:

$$D_s \equiv \frac{S_{scat}(\text{at the radar})}{\frac{P_{\text{tot}}}{4\pi r^2}}$$

For a Hertzian dipole, $S \propto \frac{\sin^2\theta}{r^2}$, $D_s\left(\theta,\phi\right) = \frac{3}{2}\sin^2\theta$.
For backscatter $\theta = 90^\circ$, so the radar cross section of one electron is

$$\sigma = 4\pi r_e^2 \approx 10^{-28}\ \mathrm{m}^2 \qquad (\sim 0.9979 \times 10^{-28}\mathrm{m}^2)$$

Why Can We Ignore the Ions?

$$\sigma_e \propto \frac{1}{m_e^2}$$

The scattering cross section of an ion is

$$\sigma_i = \frac{m_e^2}{m_i^2}\sigma_e$$

For an O^+ plasma

$$\frac{m_e^2}{m_i^2} = 1.16 \times 10^{-9}$$

Rough Dectectability Calculations

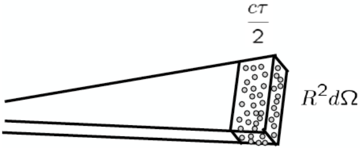
Radar Equation:

$$P_r = P_t \frac{G}{4\pi r^2} \sigma \frac{A_{eff}}{4\pi r^2}$$

For a distribution of electrons:

$$\sigma = \sigma_e N_e V \approx \sigma_e N_e r^2 \frac{c\tau}{2} \frac{4\pi}{G}$$

$$P_r \approx P_t \sigma_e N_e \frac{c\tau}{2} \frac{A_{eff}}{4\pi r^2}$$



Approximate beam solid angle:

$$d\Omega \approx \frac{4\pi}{G}$$

For $P_t = 1\ \mathrm{MW}$, $N_e = 10^{11}\ \mathrm{m}^{-3}$, $\tau = 500\ \mu\mathrm{s}$, $r = 300\ \mathrm{km}$, $A_{eff} \approx 0.6 A_{geo}$, $A_{geo} = \frac{\pi}{4} d^2$, and a dish diameter of $d = 300\ \mathrm{m}$, this gives:

$$P_r = 2.81 \times 10^{-14}\ \mathrm{W}$$

For a smaller radar with $d = 30\ \mathrm{m}$, $P_r = 2.81 \times 10^{-16}\ \mathrm{W}$

Radio Noise

Nyquist Noise Theorem: $P_N = k_B T_{sys} B$

- A good UHF receiver will have a $T_{sys} \approx 125\ \mathrm{K}$.
- B is the receiver bandwidth.

Doppler shift from electron thermal motion:

$$\Delta f = \frac{2}{c} f_{\mathrm{Tx}} v \approx \frac{2}{c} f_{\mathrm{Tx}} \sqrt{\frac{k_B T_e}{m_e}}$$

Let's assume we need to capture $B = 4\Delta f$ to get the full spectrum.
For $f_{\mathrm{Tx}} = 450\ \mathrm{MHz}$ and $T_e = 1000\ \mathrm{K}$:

$$B = 1.48\ \mathrm{MHz} \Rightarrow P_N = 2.55 \times 10^{-15}\ \mathrm{W}$$

What if instead the bandwidth is related to the ion motion?

$$v_i = \sqrt{\frac{m_e}{m_i}} v_e \Rightarrow v_i = 5.83 \times 10^{-3} v_e \text{ for } \mathrm{O}^+$$

The same numbers would yield

$$B = 8.63 \mathrm{kHz} \Rightarrow P_N = 1.48 \times 10^{-17}\ \mathrm{W}$$

Thomson Scatter Summary

- Thomson scatter from electrons is a fundamental physical process
- Radar cross section of one electron is a constant independent of wavelength ($\sim 10^{-28}\ \mathrm{m}^2$)
- Scatter from ions is negligible
- Even though one electron has a tiny cross section, scatter can still be detectable from a whole volume of electrons