IS Fitting Examples, Ambiguities, and Constrained Analysis

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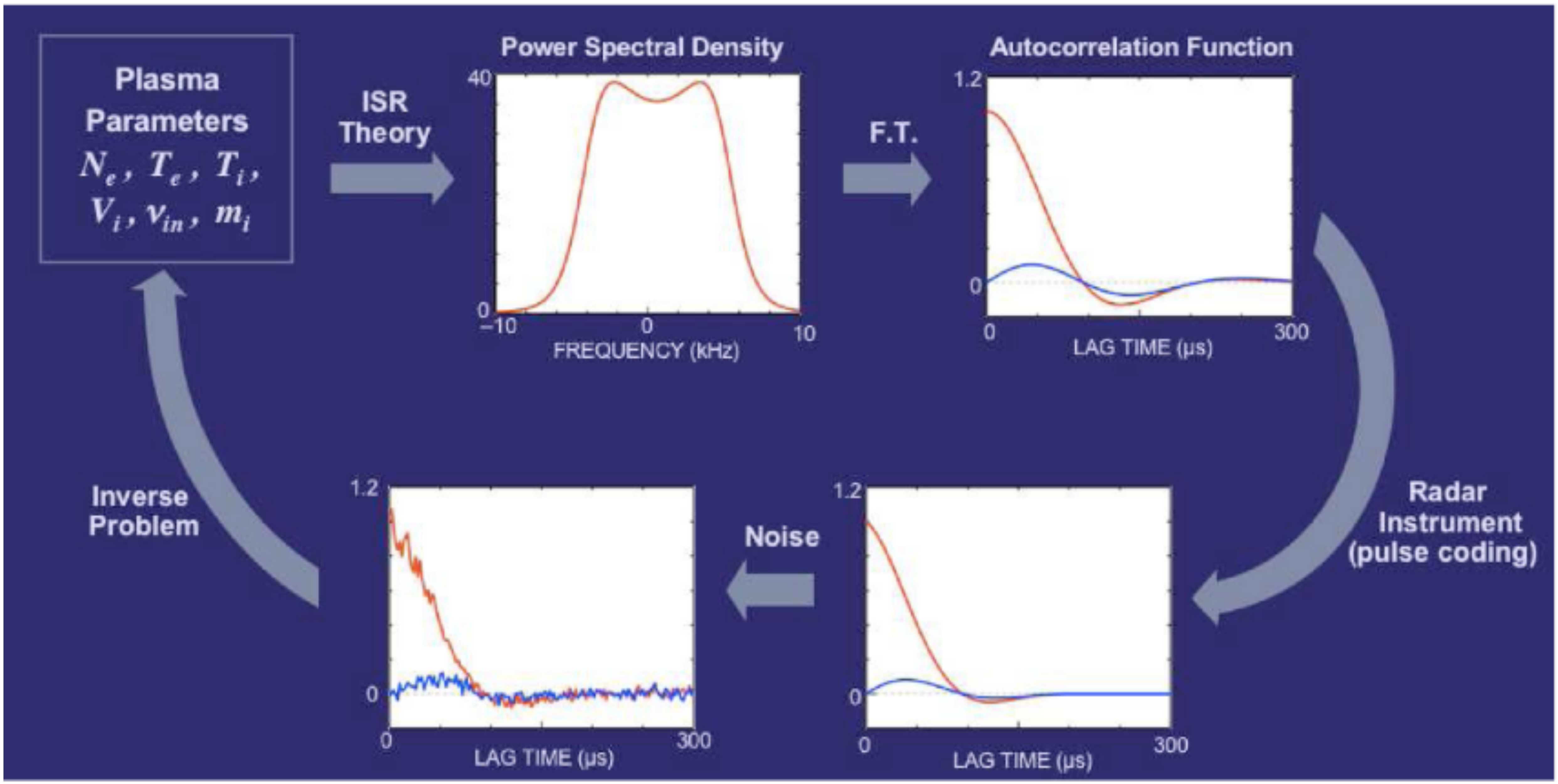
Topics covered:

- Fitting data to a model
- Effects of noise and forward model ambiguities
- Resolving IS fitting ambiguities: constrained analysis

ISR Practicalities: Data Reduction

NB: Power spectrum (freq domain) <-> Autocorrelation function (time domain)



Fitting data to a model

Goal: minimize $$\chi^2(\mathbf{p}) = \sum_{m=1}^{M} \frac{[z_m - f_m(\mathbf{p})]^2}{\sigma_m^2}$$ "L2 Norm"

- data → z_m
- Forward model → f_m
- Parameter vector → $\mathbf{p}$
- Measurement uncertainties → σ_m^2

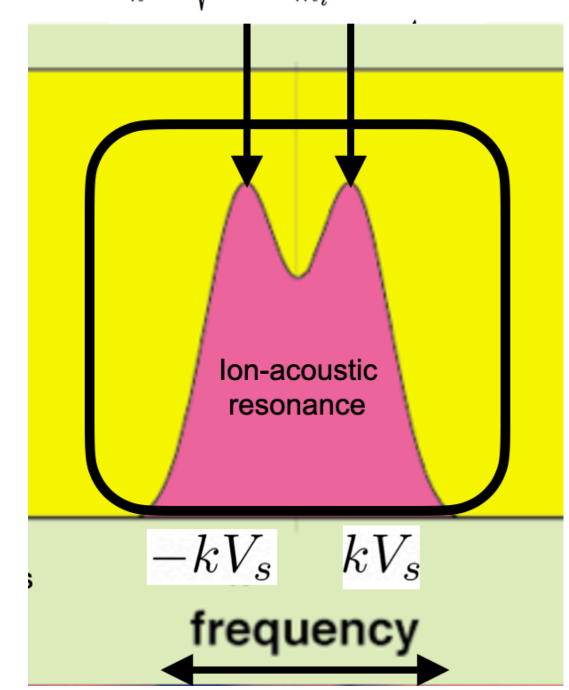
Minimize by iterating over parameter vector **p**.

Some problems are linear least-squares: solvable in one step.
Others are nonlinear least-squares:
model has complicated variations with parameters.
Incoherent scatter is this type.

Many different fitting algorithms possible depending on how one analytically expands the minimization function:

- Gradient-search (Nelder-Mead simplex)
- Analytic expansion (parabolic surface)
- Levenberg-Marquardt (balance between gradient and analytic)
- Simulated annealing

$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = V_s$$

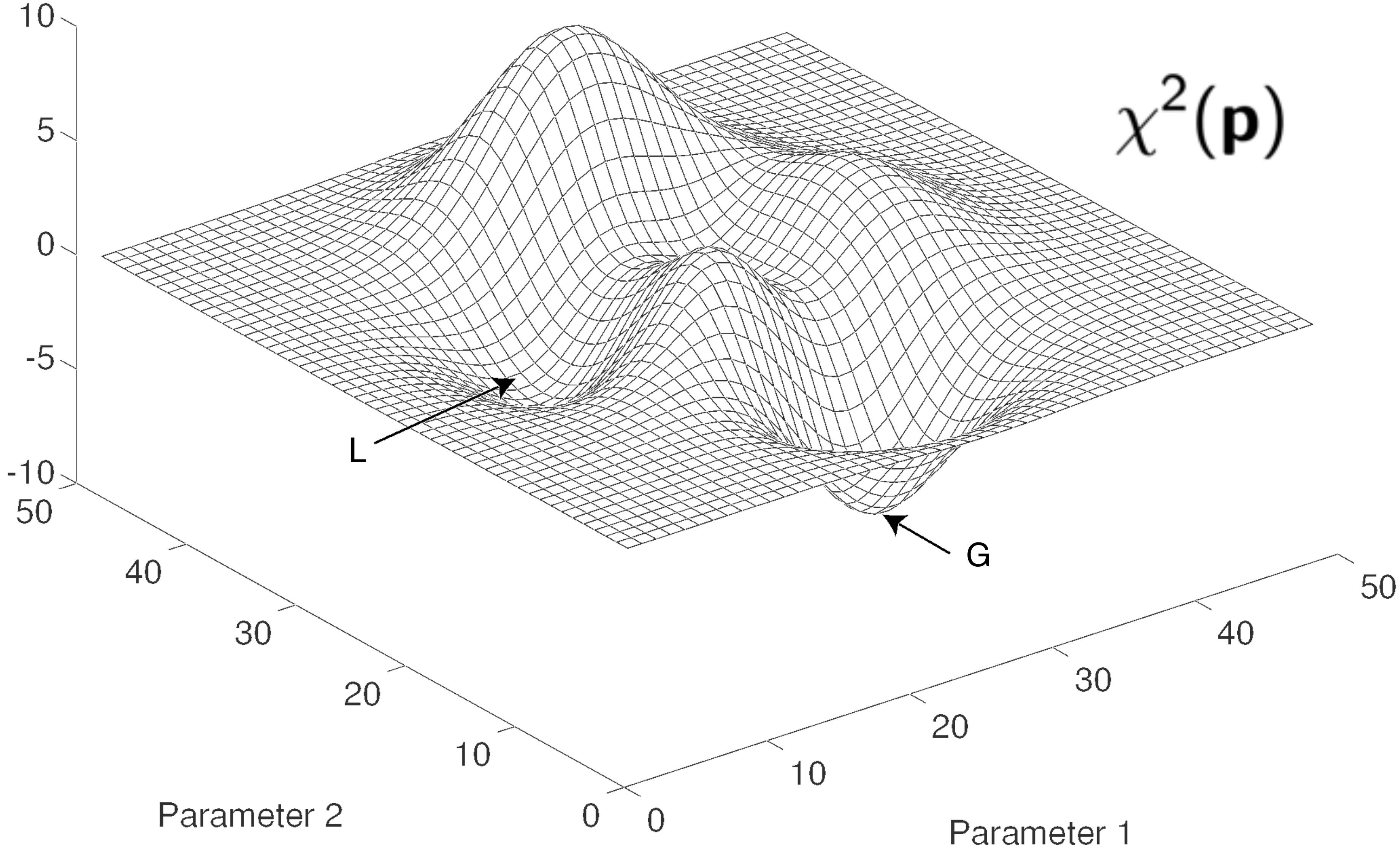


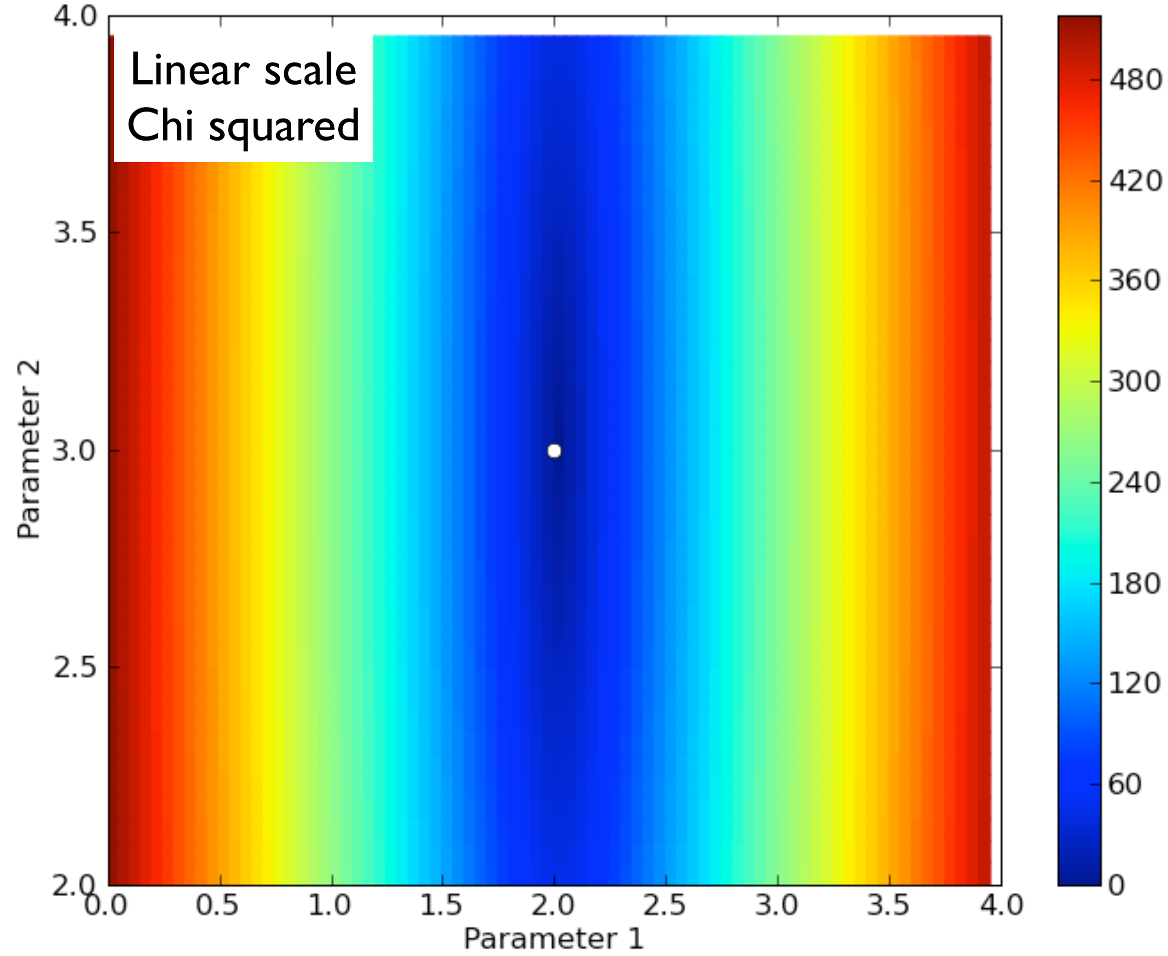
$f_m(\mathbf{p})$

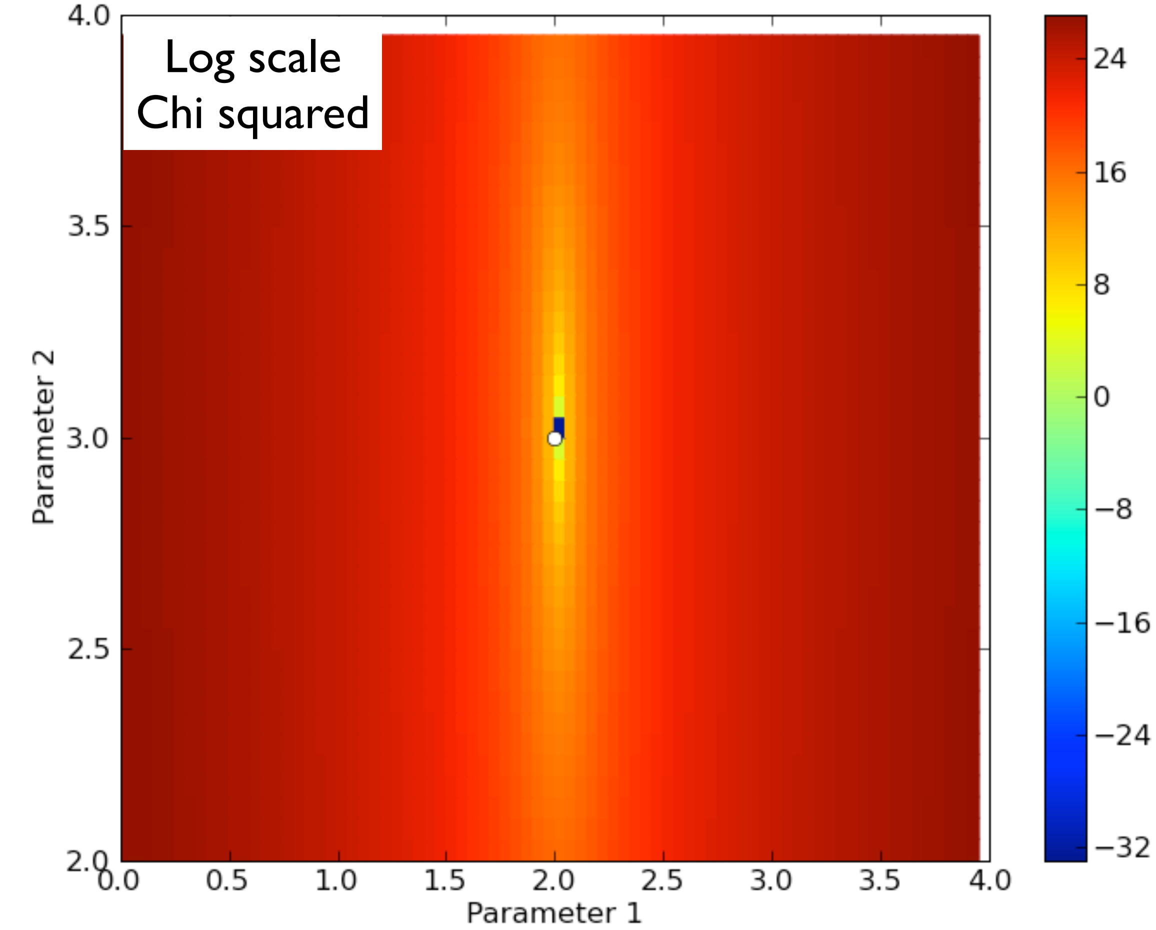
$\mathbf{p} = [N_e, T_e, T_i, m_i]$

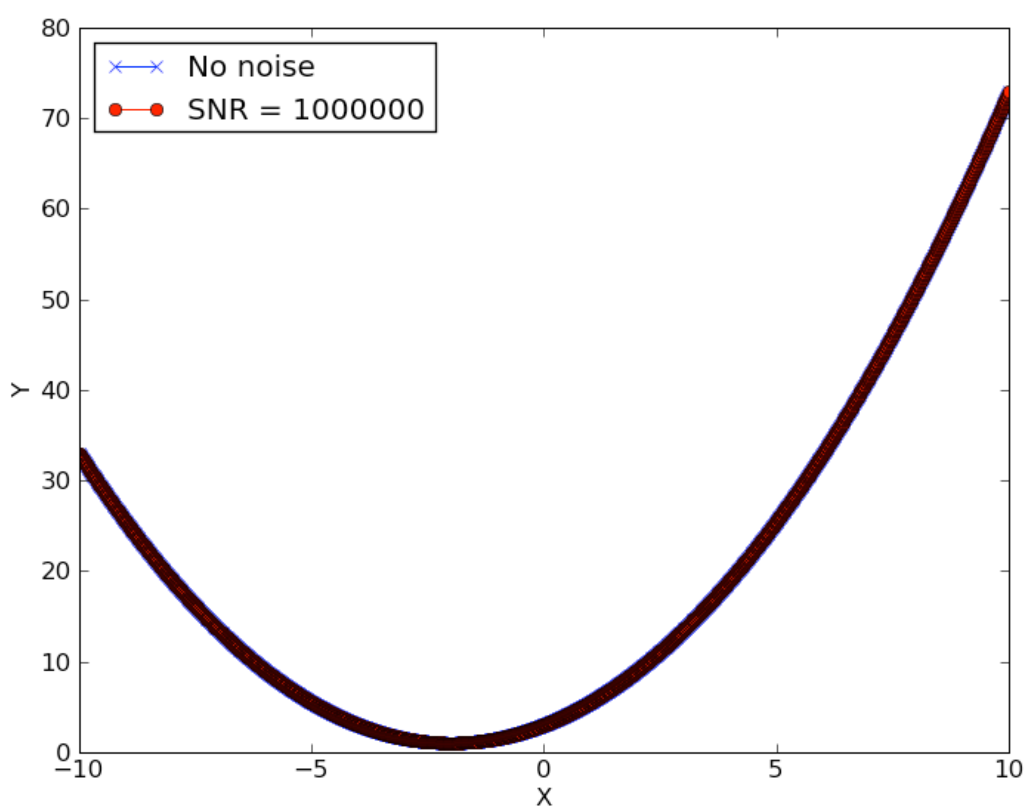
Incoherent Scatter Fit Ambiguities

L, G might both be valid parameter solutions. (**Local** vs **global** minimum)
Might need to use constraints on the parameters to decide which one.







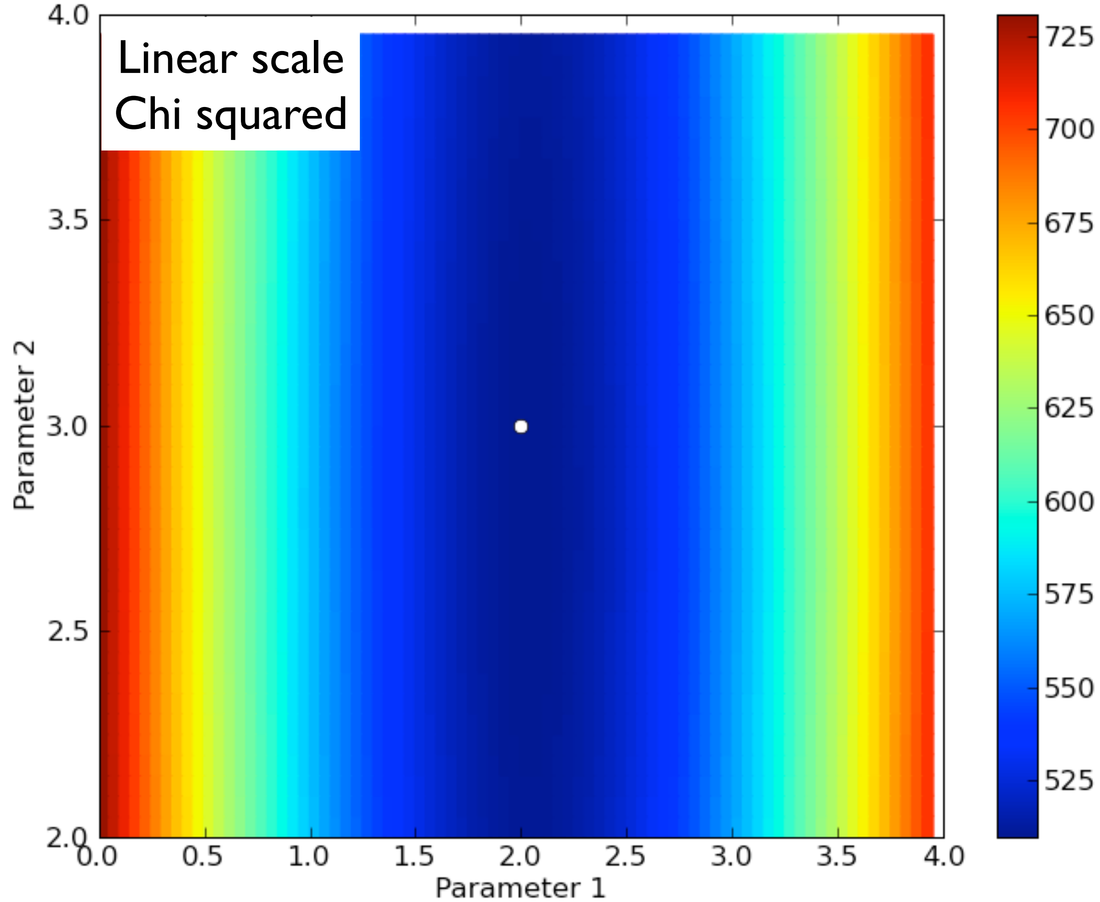


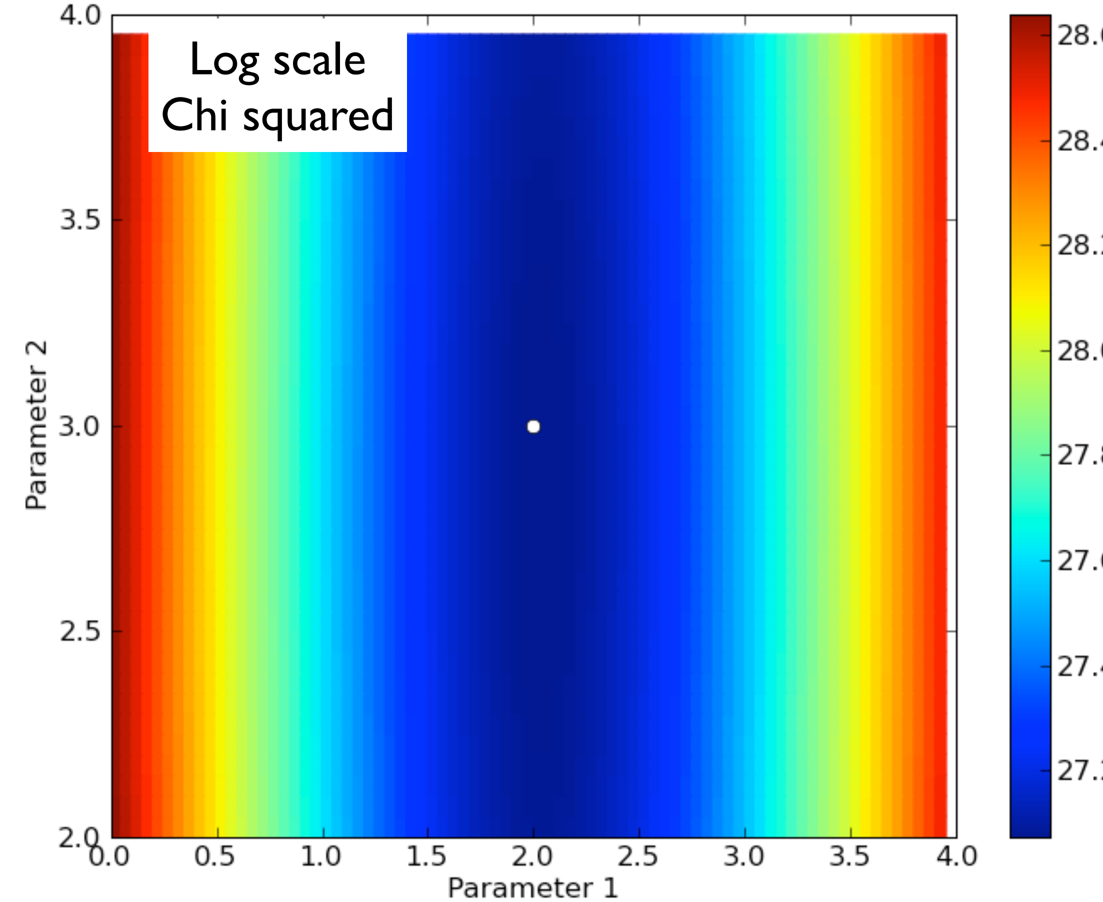
Parabola

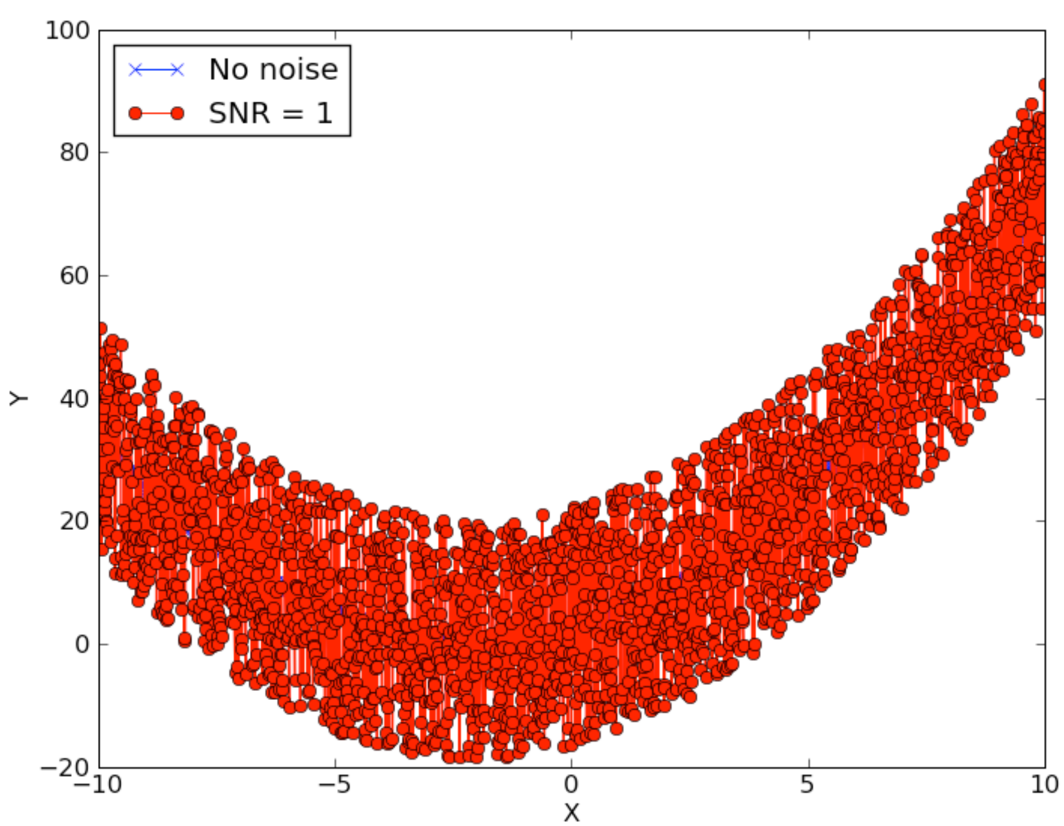
$y = 0.5\ x^2 + 2\ x + 3$

Slope, intercept fit

No noise







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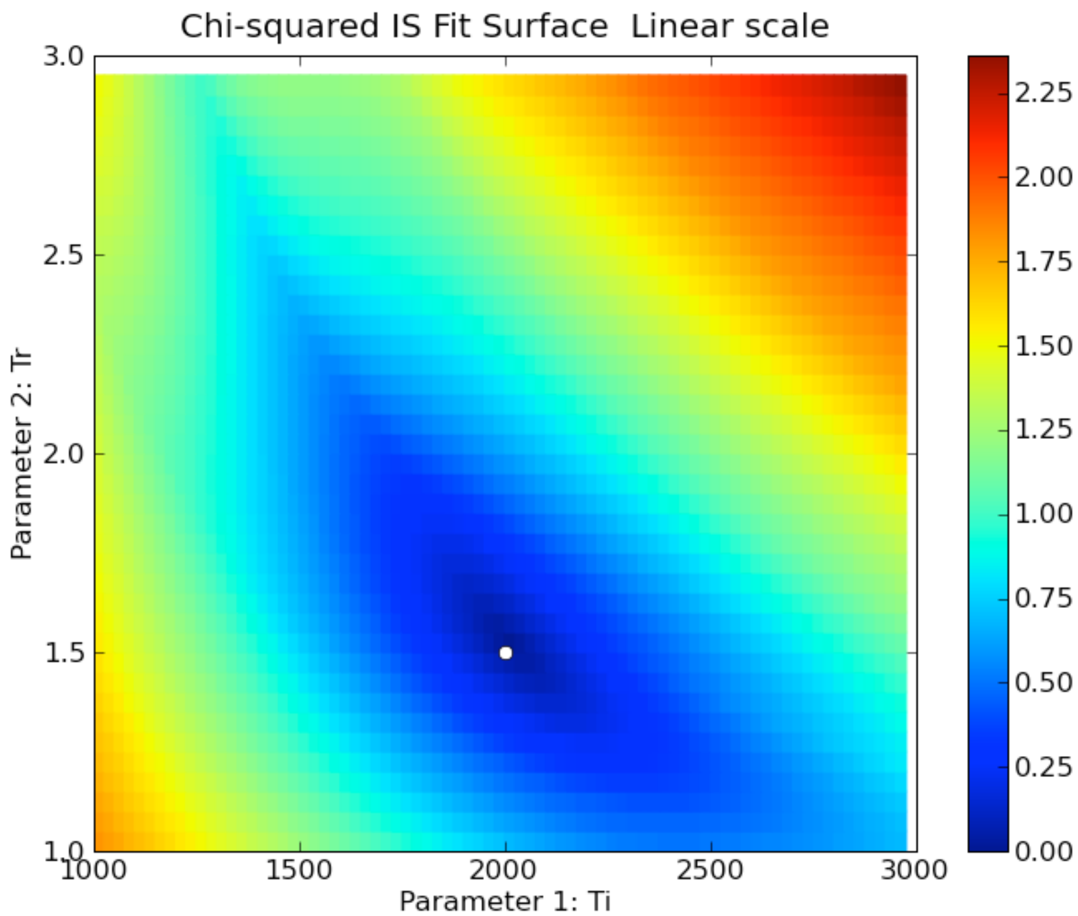
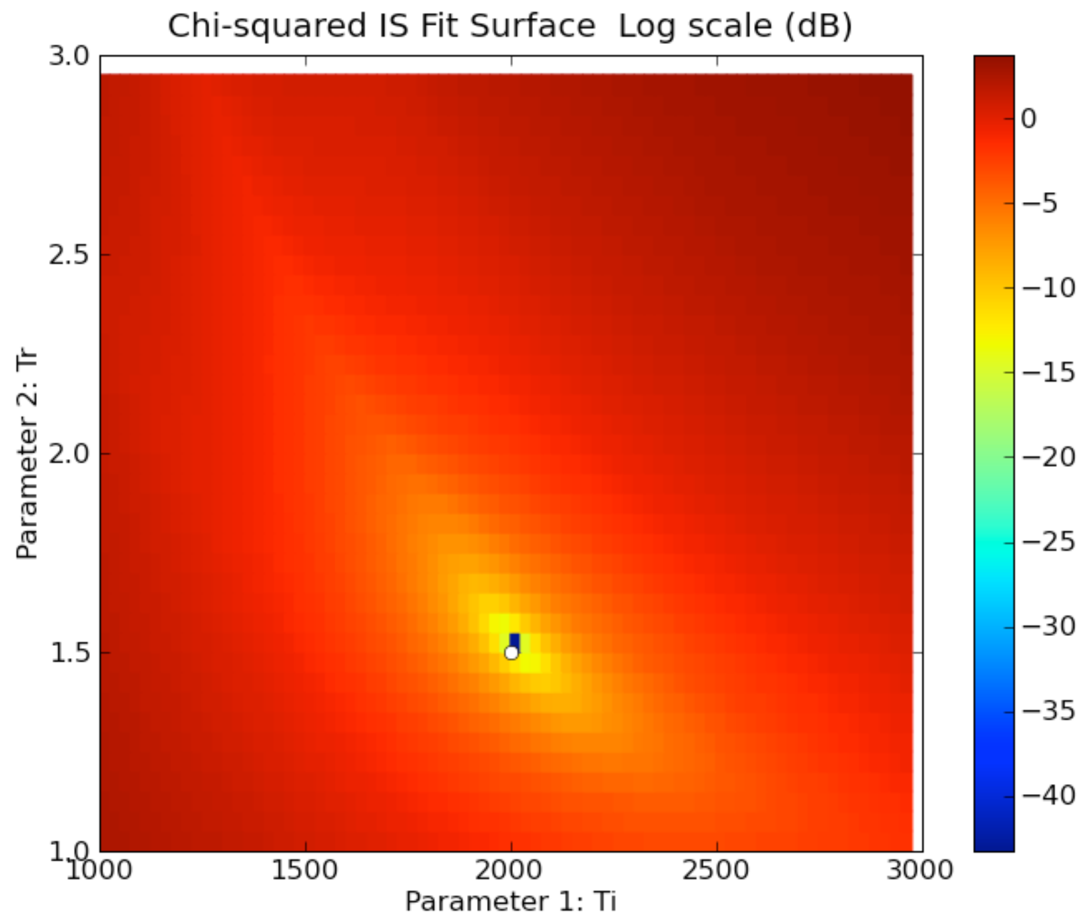
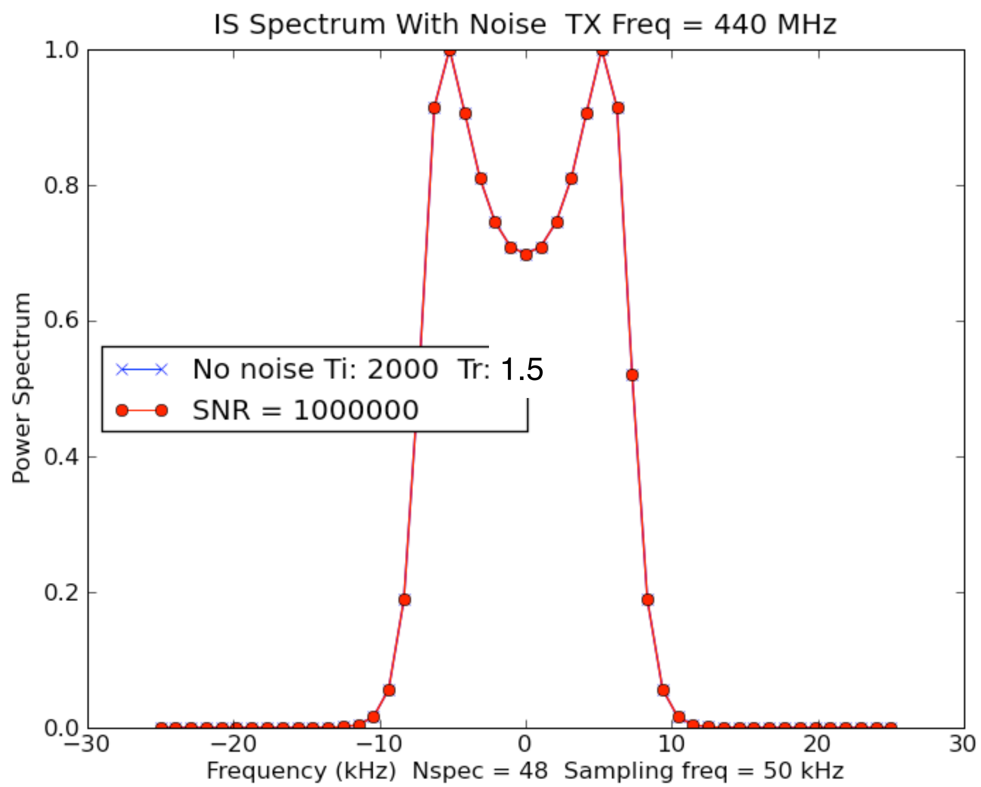
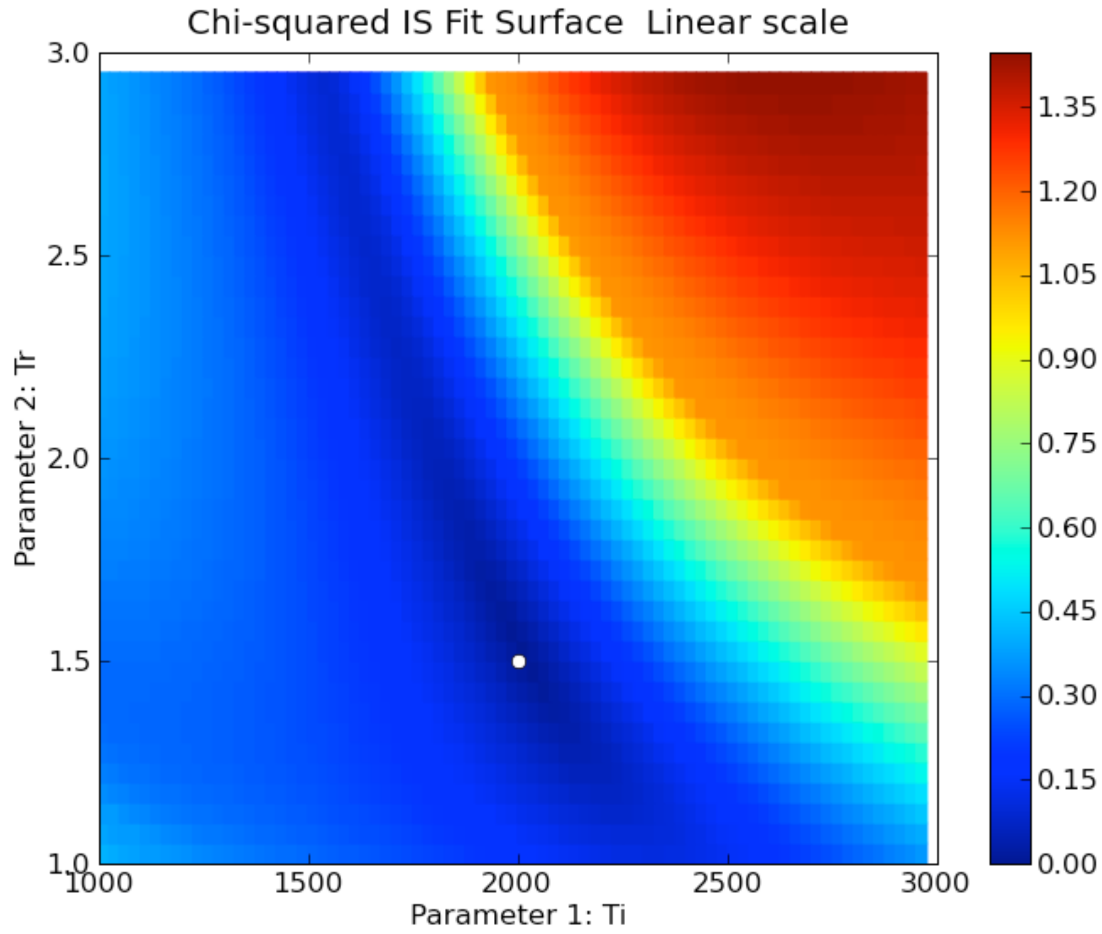
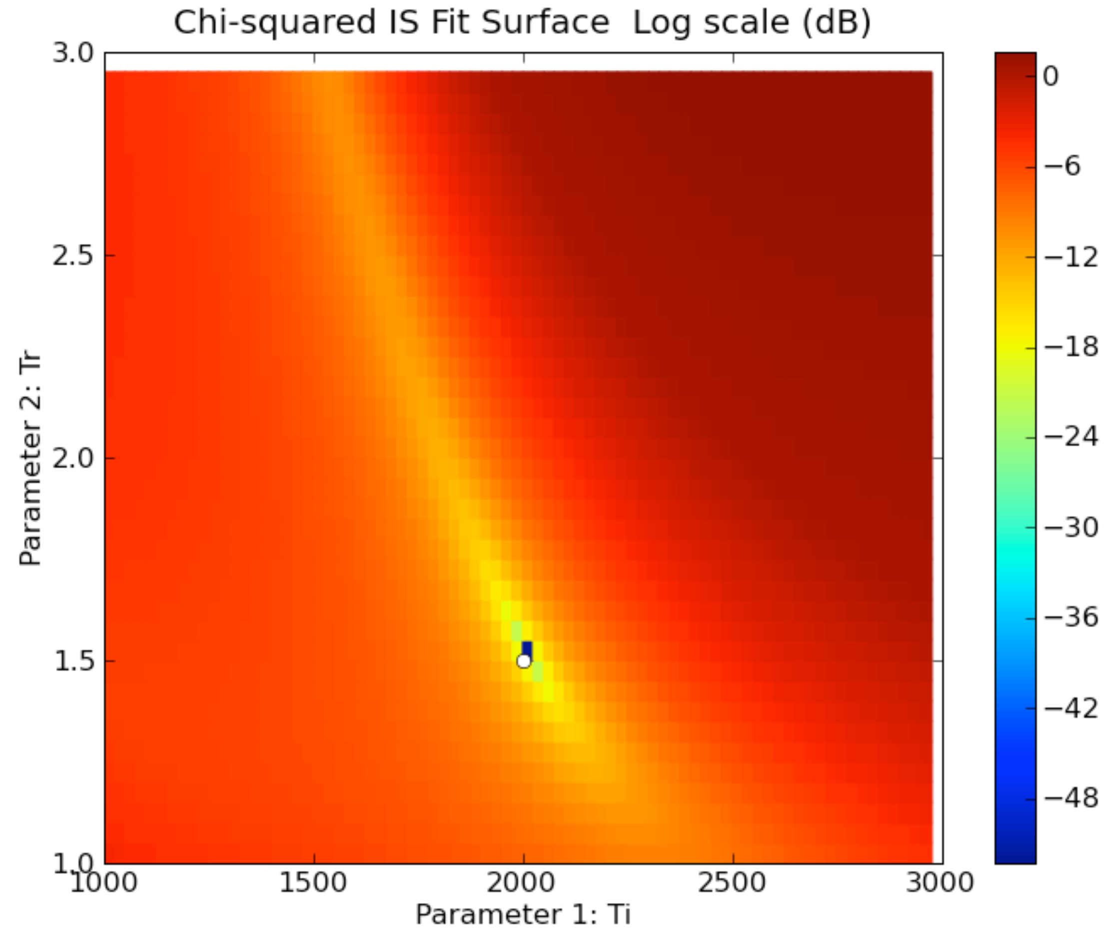


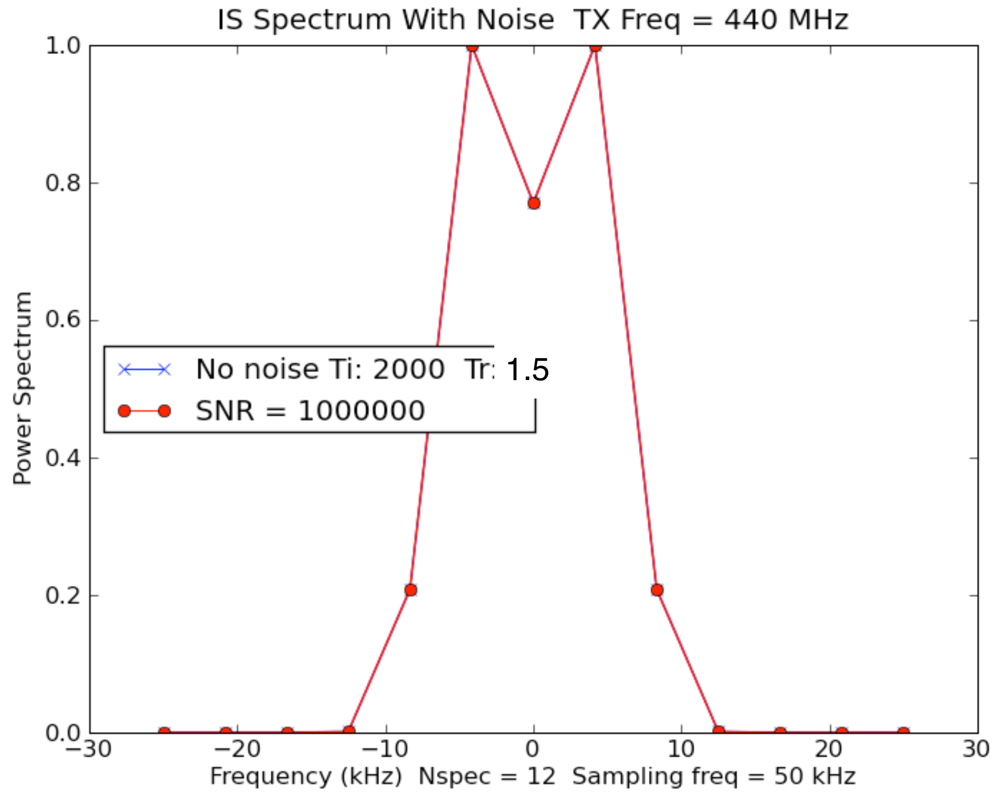




440 MHz IS Spectrum
Ti/Tr space
Ti = 2000 Tr = 1.5
No noise







440 MHz IS Spectrum

Ti/Tr space

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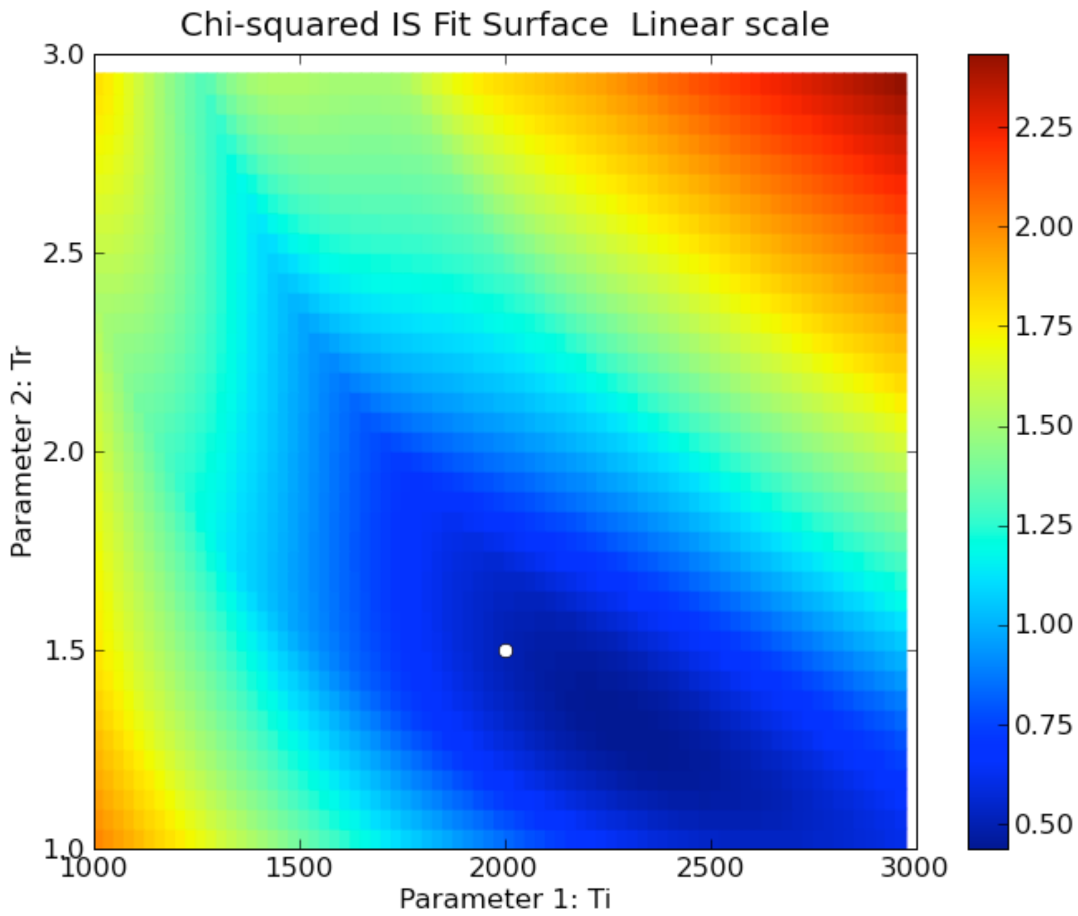
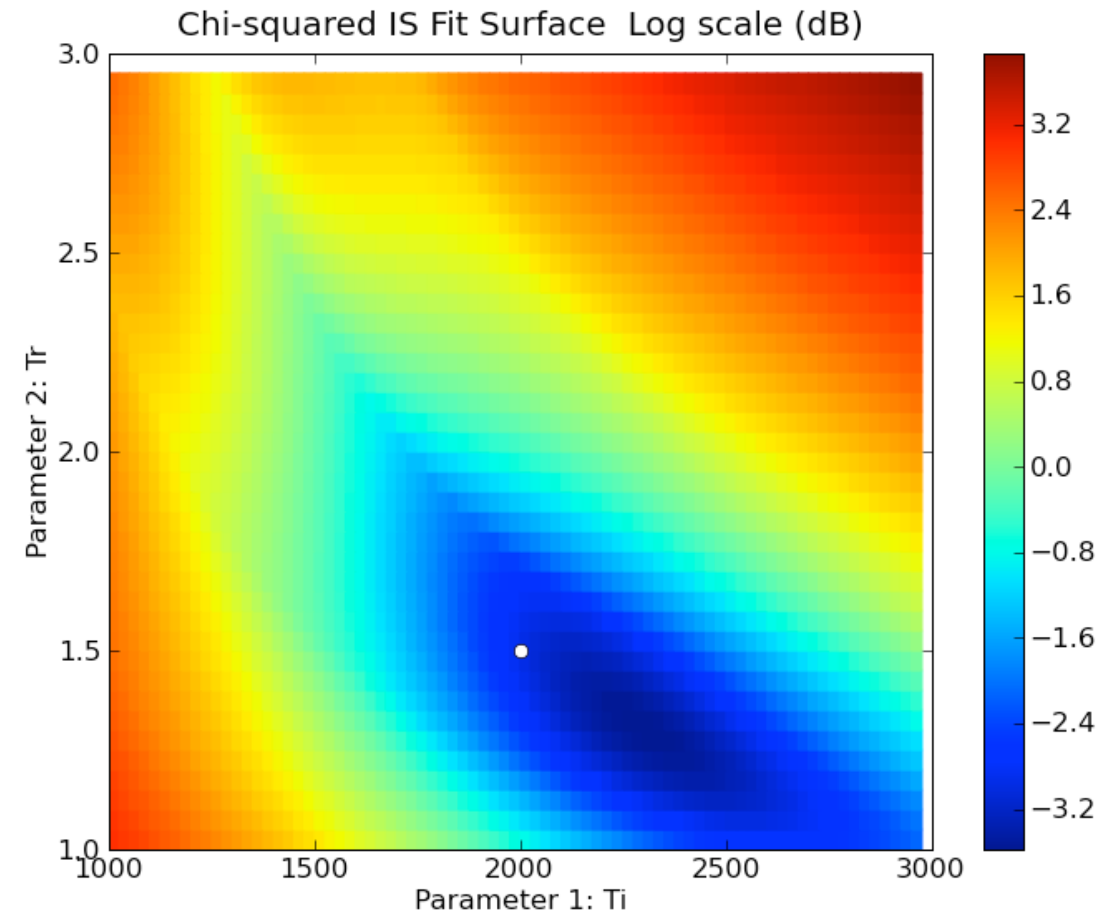
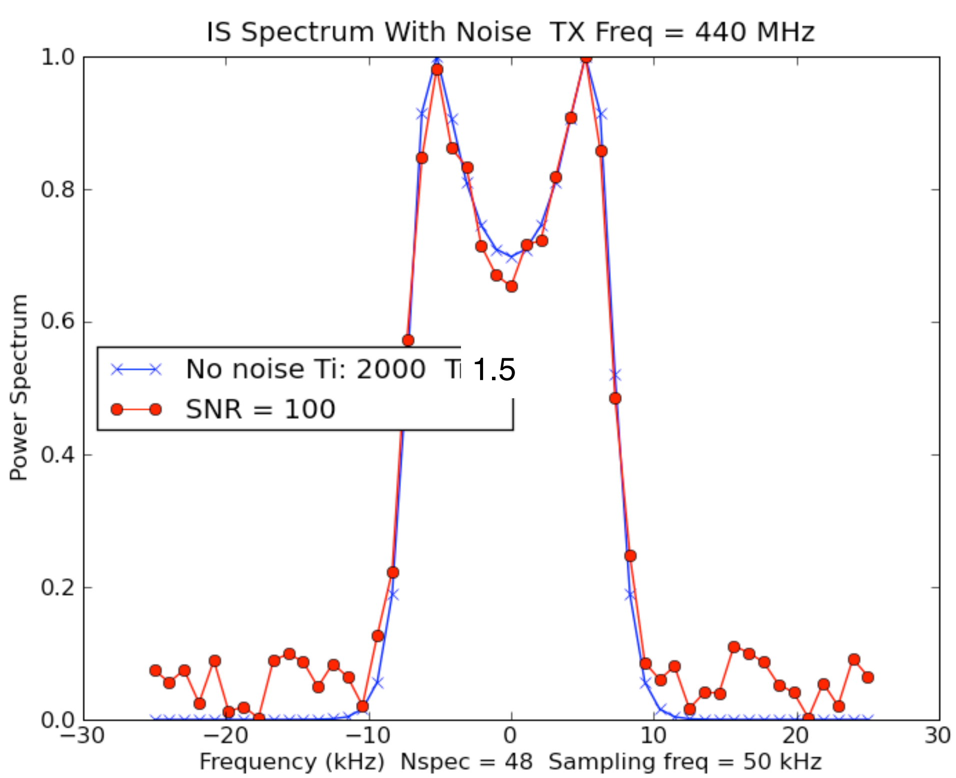
Poor sampling







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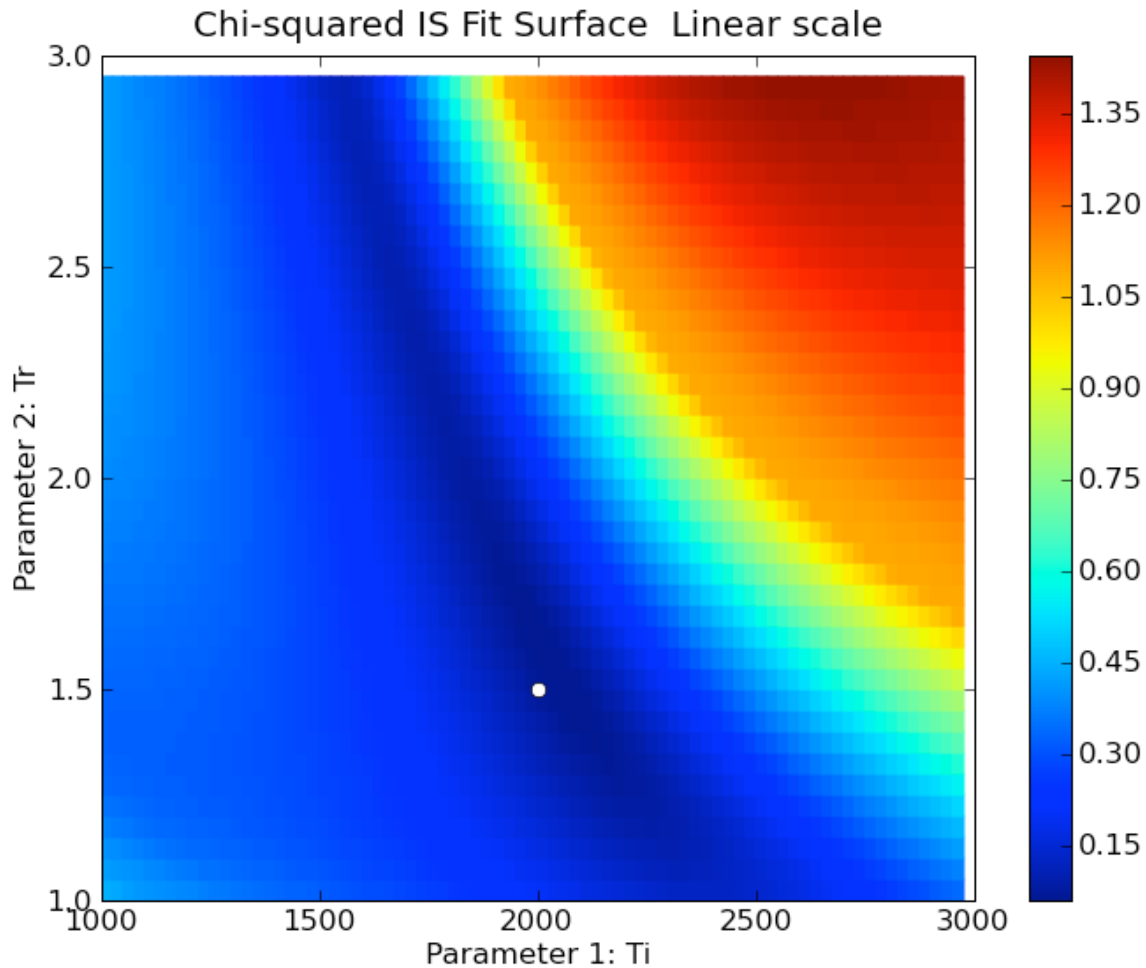

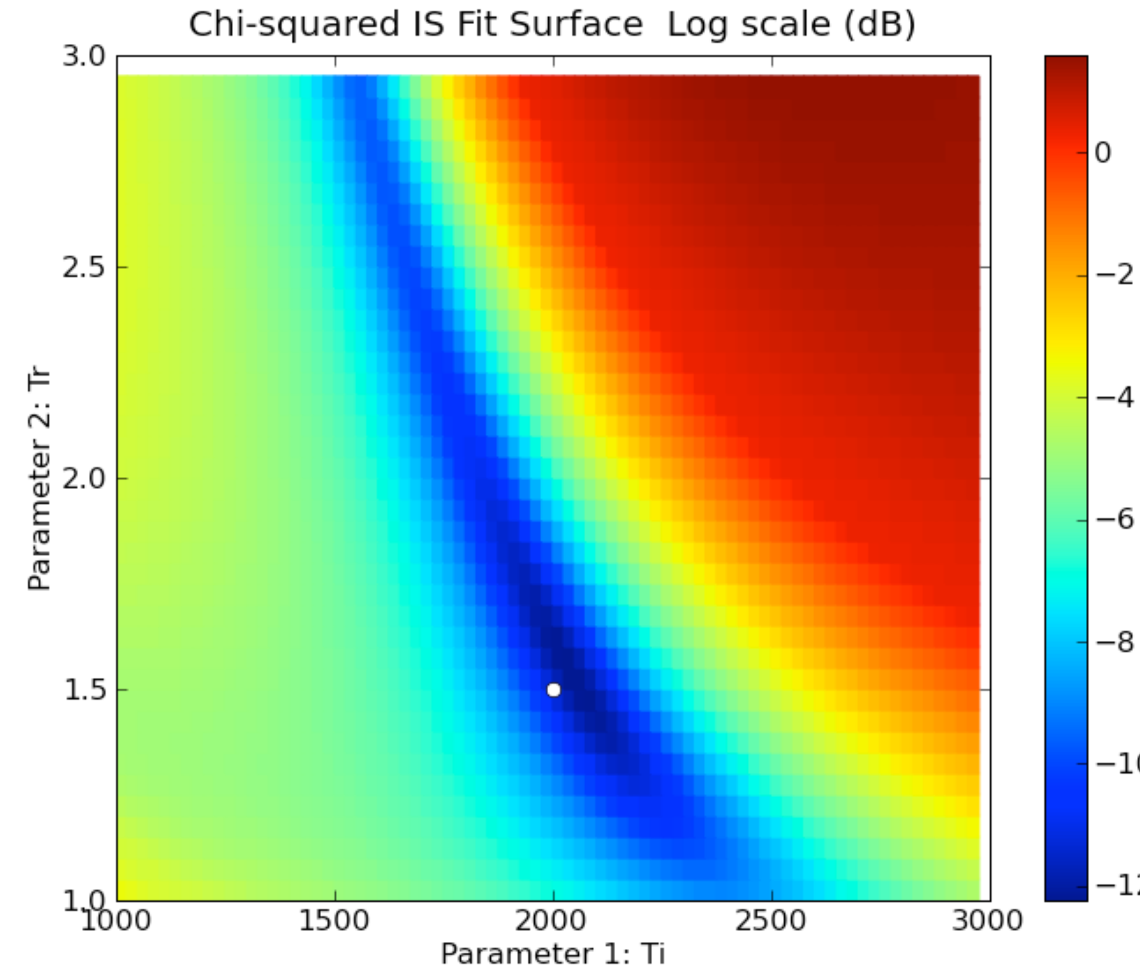

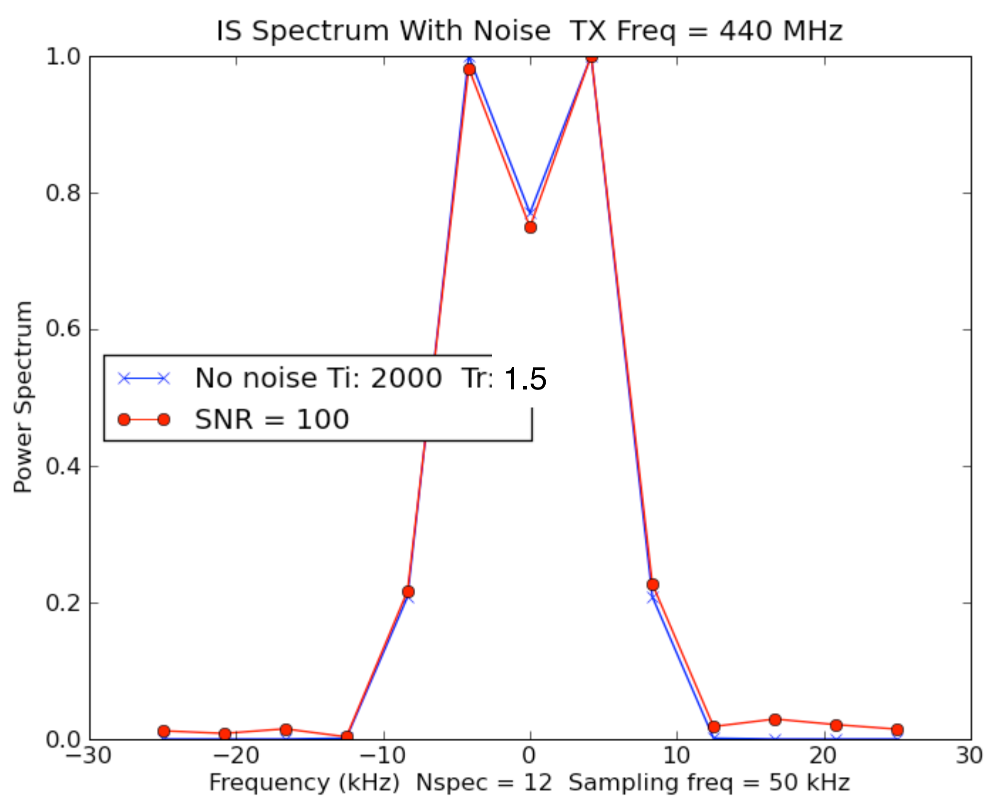

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Noisy, poor sampling

Eigenvalues of Hessian matrix (2nd derivative of min fn) has insights on parameter ambiguities

Table 5.1: Fit results and uncertainty values at 923 km for conditions over Arecibo at 20:41 LT on January 14, 1991. The most ill-defined parameter vector is found from the Hessian matrix eigenvector with the smallest eigenvalue.

$$\hat{\beta}_{\mathrm{LS}} = \min_{\beta} \sum_i \frac{[h_i(\beta) - Z_i]^2}{\sigma_i^2}$$

$$Cov\left\{\hat{\beta}_{\mathrm{LS}}\right\} \approx \left[\tilde{J}^T \tilde{J}\right]^{-1}$$

$$\tilde{J} = \begin{pmatrix} \frac{1}{\sigma_0}\frac{\partial h_0}{\partial \beta_0} & \frac{1}{\sigma_0}\frac{\partial h_0}{\partial \beta_1} & \cdots & \frac{1}{\sigma_0}\frac{\partial h_0}{\partial \beta_{M-1}} \\ \frac{1}{\sigma_1}\frac{\partial h_1}{\partial \beta_0} & \frac{1}{\sigma_1}\frac{\partial h_1}{\partial \beta_1} & \cdots & \frac{1}{\sigma_0}\frac{\partial h_1}{\partial \beta_{M-1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sigma_{N-1}}\frac{\partial h_{N-1}}{\partial \beta_0} & \frac{1}{\sigma_{N-1}}\frac{\partial h_{N-1}}{\partial \beta_1} & \cdots & \frac{1}{-}\frac{\partial h_{N-1}}{\partial \beta_{M-1}} \end{pmatrix}$$

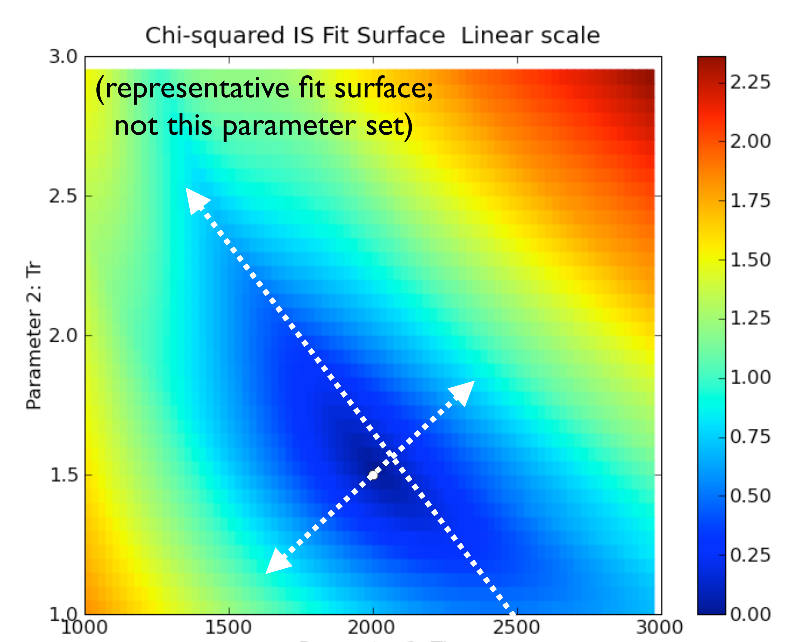
Fitter results:

	Best-fit results	Uncertainty
N_e	6.41×10^4	7.39×10^3
T_e	2285	41.3
T_i	2223	23.4
f_{H^+}	0.490	0.00483
f_{He^+}	0.159	0.00341

Correlations between pairs of parameters:

Param Pair	Correlation	Param Pair	Correlation
$[N_e, T_e]$	0.958	$[T_e, f_{H^+}]$	-0.916
$[N_e, T_i]$	-0.649	$[T_e, f_{He^+}]$	-0.401
$[N_e, f_{H^+}]$	-0.849	$[T_i, f_{H^+}]$	0.414
$[N_e, f_{He^+}]$	-0.511	$[T_i, f_{He^+}]$	0.515
$[T_e, T_i]$	-0.440	$[f_{H^+}, f_{He^+}]$	0.099

Eigenvalues of $\left[\tilde{J}^T \tilde{J}\right]$



Most ill-defined parameter combination:

$+0.998\ (N_e)\ +0.0527\ (T_e)\ -0.0202\ (T_i)\ +\ 5.46 \times 10^{-6}\ (f_{H^+})\ -\ 2.32 \times 10^{-6}\ (f_{He^+})$

Improving the fit: adding priori constraints

Bayesian statistics: add apriori knowledge to stabilize fit.

Can come from other instruments, or from data at other altitudes/times.

One formulation: minimize

$$\chi^2 = \chi^2_{data} + \chi^2_{apriori}$$

Here, the apriori information adds a cost for solutions which wander too far from the apriori knowledge. (DANGER!)

Many implementations in our field:

Constrained temperature profiles

Vector velocity fits

Full profile analysis

Regularization

Etc.

(techniques such as Tikhonov regularization used to set relative weight of data and apriori terms when minimizing)

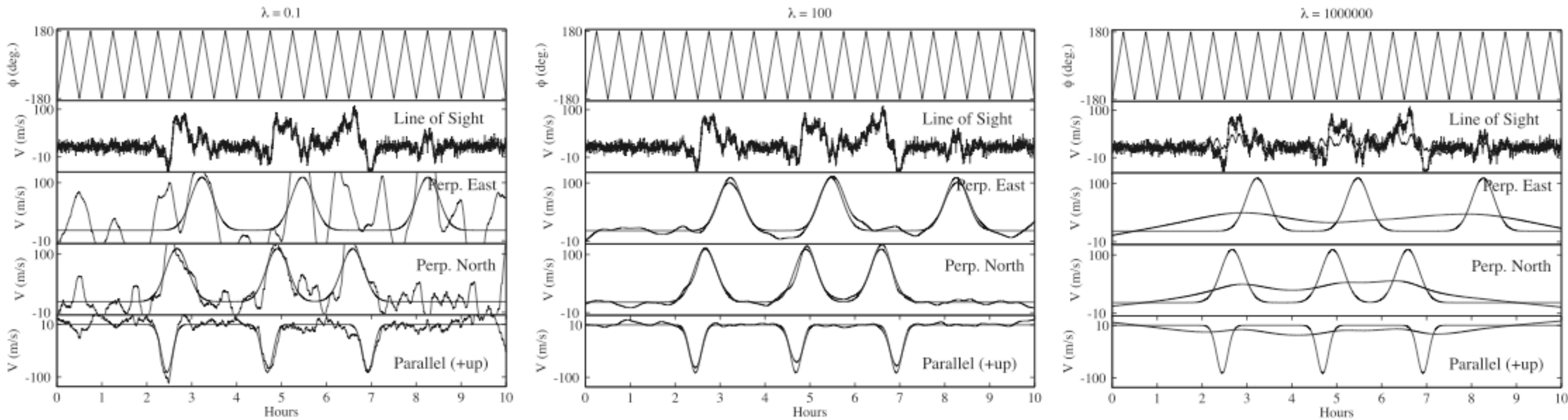


Figure 3. Vector velocity input-output comparison using a simulation assuming a single beam and applying the method of regularization. The panels on the left show the results for a small value of λ. The panels on the center were obtained from a simulation with an optimal value of λ, while the panels on the right correspond to a case with too much λ.

$$\begin{bmatrix} V_{pn} \\ V_{pe} \\ V_{par} \end{bmatrix} = \begin{bmatrix} -\cos\delta\sin I & \sin\delta\sin I & \cos I \\ \sin\delta & \cos\delta & 0 \\ \cos\delta\cos I & -\sin\delta\cos I & \sin I \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}.$$

$$\begin{bmatrix} V_{LOS}(1) \\ \vdots \\ V_{LOS}(n) \end{bmatrix} = \begin{bmatrix} -\cos\phi_1\sin\theta & \sin\phi_1\sin\theta & \cos\theta \\ & \vdots & \\ -\cos\phi_n\sin\theta & \sin\phi_n\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Arecibo linear regularization of line-of-sight velocities for full vector derivation

Sulzer et al, 2005

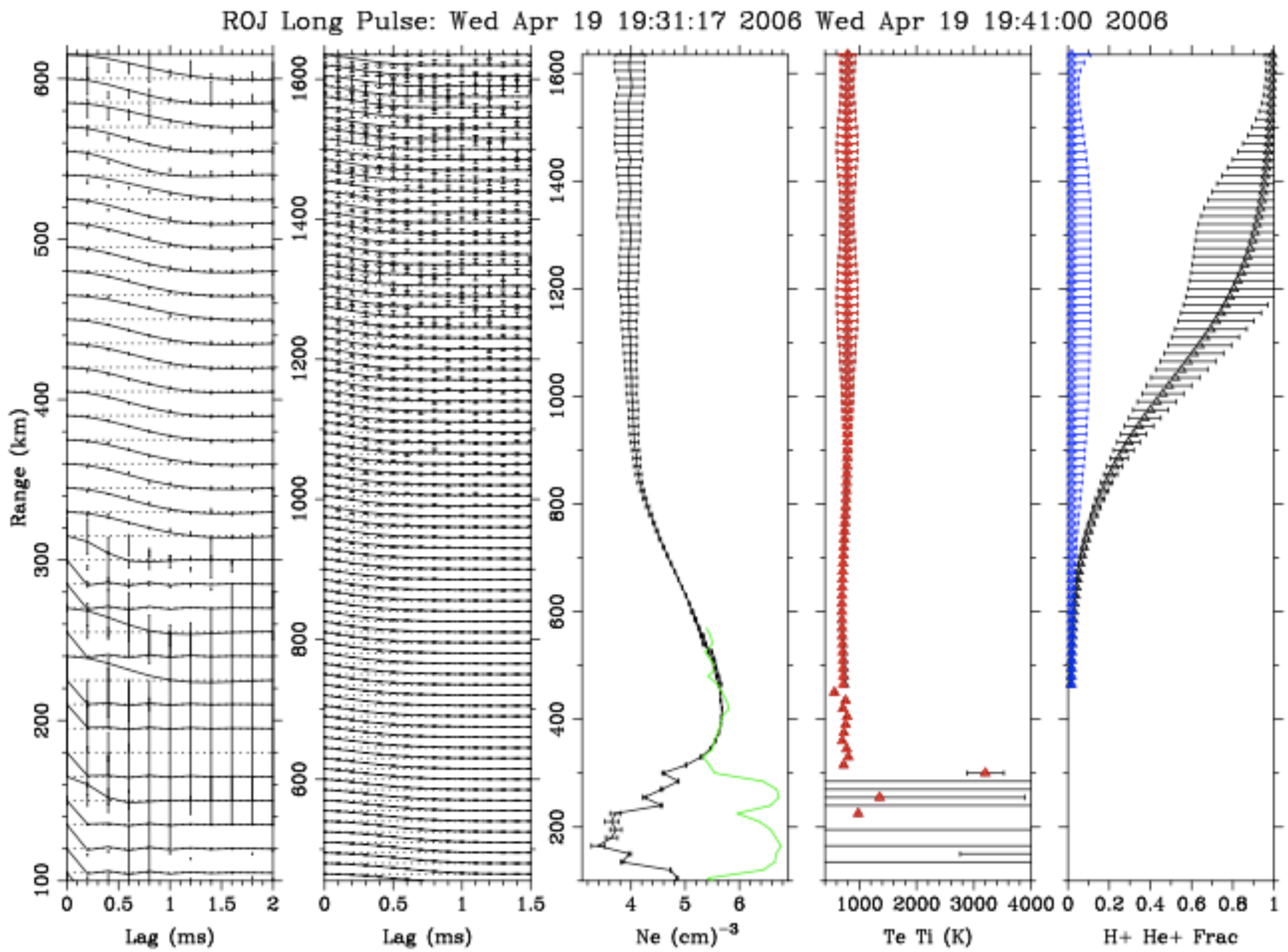


Fig. 3. Jicamarca profiles for 19:30 LT (00:30 UT). From left to right, the panels represent double-pulse lag products, long-pulse lag products, electron density, electron and ion temperature, and light ion fraction (see text).

Full profile at JRO Hysell et al, 2008
6 cost functions inject weighted apriori information

IS Fitting Examples, Ambiguities, and Constrained Analysis

Summary:

- Fitting forward model to data yields estimates of model parameters
- Many models (including incoherent scatter) can have more than one parameter solution that fits data
- Regularization or constraints can help select a final solution when carefully applied
- Consult your local friendly Geospace Facility scientist for advice