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Individual Technical Deliverable
2.980 Sports Technology

PROJECT DESCRIPTION

Foam materials are used in sports equipment to provide cushioning, energy absorption, and protection. In footwear, foam is typically used to provide cushioning, balanced with energy return. Different materials or densities are used to provide differences in mechanical properties -- like bending stiffness or compression -- while balancing the tradeoff in weight and cushioning. One can also introduce stiff plates (such as TPU or Carbon Fiber) at various positions in the foam composite to alter the impact absorption of the structure.

LITERATURE REVIEW

When speaking of running shoes, there are a couple different main parts that make up the shoe. The first is the upper, which is the part of the shoe that covers the individuals foot. Second is the outsole, or the bottommost part of the shoe that directly comes in contact with the ground. Lastly is the midsole, which lies between the upper and the outsole and is the most important part of the shoe when it comes to comfort, injury prevention, and running performance [AAPSM]. For this specific project, our team will be focusing only on the midsole of the shoe.

Shoe midsoles are mainly made of a closed cell foam material: the most common types being thermoplastic urethane (TPU) and ethyl vinyl acetate (EVA) [AAPSM]. Stiff plates may also be incorporated in the midsole in hopes to improve overall shoe performance [Gear Patrol]. All of the leading shoe companies such as Nike, Adidas, Asics, New Balance, and Under Armour all have their own foam and plate technologies that they believe to be best [Forbes].

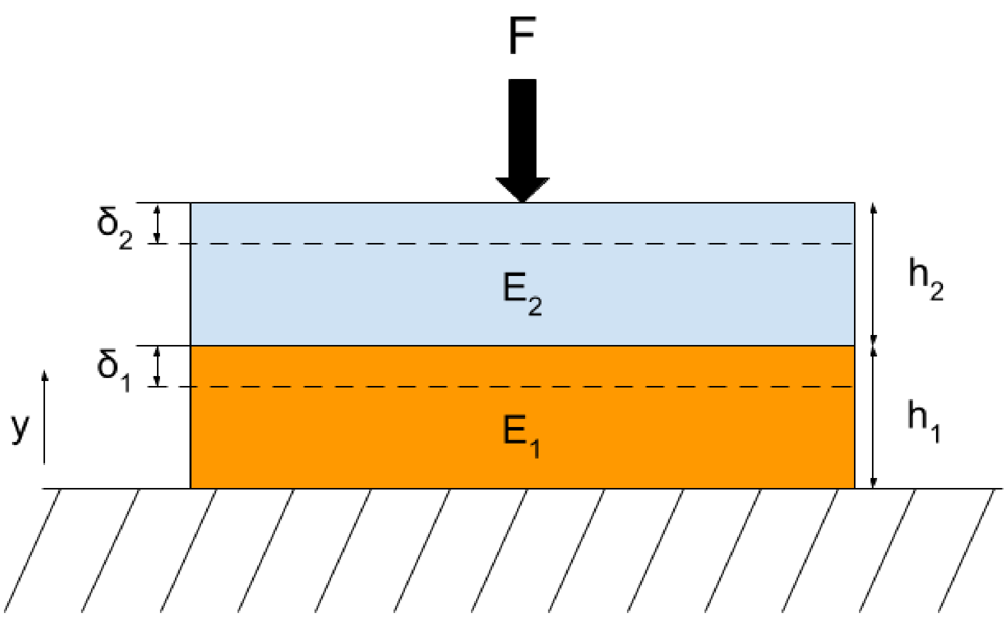
The mechanical behaviors of these different foam materials cannot be modeled using a perfectly linear elastic model. Instead, more complex material properties are required to accurately produce a model for the load-compression behavior of these foams. These properties include: E_0 (the Young's Modulus of the base polymer), φ (the volume fraction of the polymer), and a dimensionless function of the compressive strain experienced in the polymer, $\psi(\varepsilon)$. This function is determined by experimental load-compression data. Using these properties and certain relations between them, it is possible to determine what material properties a foam must have to pass a certain load requirement [Rusch].

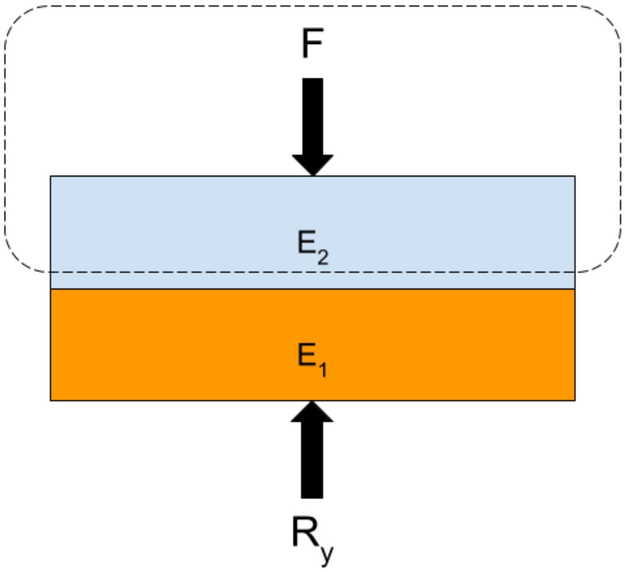
One of the most important aspects of a running shoe is minimizing the loss of energy during a run, and midsole material properties / geometry play a large role in retaining energy. If lighter materials are used, there is less energy required for the runner to accelerate / decelerate the shoe. This weight factor plays a very significant role in events where the runner is sprinting, and speeds of up to 20 m/s can be reached in very short amounts of time. Using certain midsole materials can also decrease the energy required to limit the impact forces as a runner's foot strikes the ground. Midsoles can also decrease the amount of energy used to stabilize certain joints in a runners leg [Stefanyshyn]. Adding stiff plates to the shoe midsole can also reduce energy lost as a runners big toe bends, and can redistribute impact forces during runs [The Conversation].

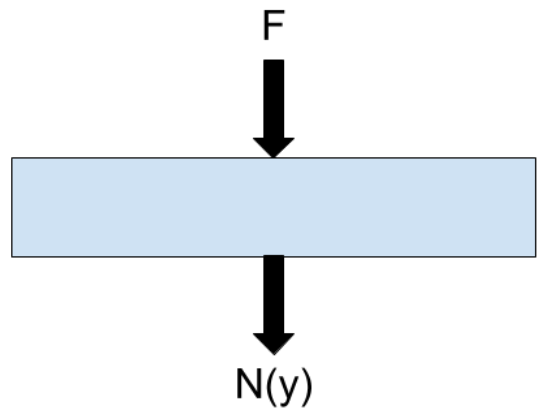
Although there is currently some strong information regarding minimizing energy lost, there are many different running styles, and foot / leg anatomies, so there cannot be a “one shoe fits all design.” Many runners are looking for a shoe that specifically compliments their running style and body. This desire somewhat validates the objective of our groups project, as we are interested in creating a model for a shoe midsole that better matches an athletes ideal shoe.

MATHEMATICAL EXPRESSIONS

COMPRESSION







Equilibrium:

$$\Sigma F_y = 0 \rightarrow F = R_y$$
$$N(y) = -F$$

Constitutive Relationships:

$$E_2 * \varepsilon_{yy2} = \sigma_{yy}\ (top\ half)$$
$$E_1 * \varepsilon_{yy1} = \sigma_{yy}\ (bottom\ half)$$

Compatibility:

$$\varepsilon_{yy2} = \frac{\delta_2}{h_2}$$
$$\varepsilon_{yy1} = \frac{\delta_1}{h_1}$$
$$\delta = \delta_1 + \delta_2$$
$$A * \sigma_{yy} = N(y) = -F$$
$$A * E_2 * \varepsilon_{yy2} = -F$$

$$A * E_2 * \frac{\delta_2}{h_2} = -F$$
$$\delta_2 = \frac{-F*h_2}{A*E_2} \rightarrow \delta_1 = \frac{-F*h_1}{A*E_1}$$
$$\delta = \frac{-F}{A} * \left(\frac{h_1}{E_1} + \frac{h_2}{E_2}\right)$$
$$\frac{-F}{A} = \frac{\delta}{(h_1+h_2)} * \frac{(h_1+h_2)}{\left(\frac{h_1}{E_1}+\frac{h_2}{E_2}\right)}$$

Effective Compression Modulus:

$$E_{eff} = \frac{(h_1+h_2)}{\left(\frac{h_1}{E_1}+\frac{h_2}{E_2}\right)}$$

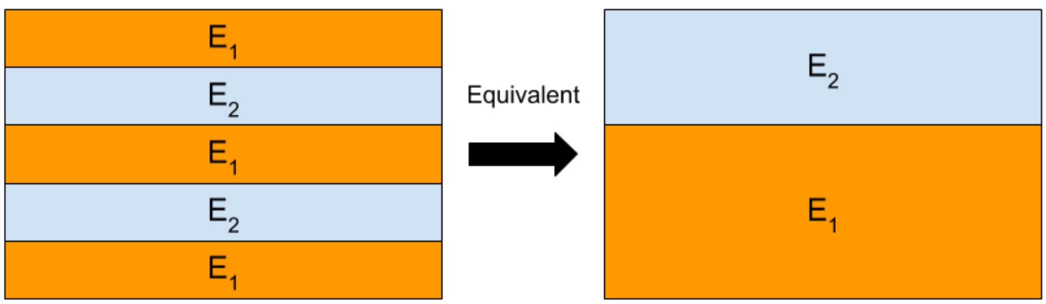
$$E_{eff} = \frac{1}{(\frac{h_1}{E_1 * h} + \frac{h_2}{E_2 * h})}$$
$$where\ h = h_1 + h_2$$

Above example is only for a two layer composite, when working with a stack of materials with two known moduli the effective modulus becomes:

$$E_{eff} = \frac{1}{\sum_i \frac{1}{E_i} * \frac{h_i}{h}}$$
$$where\ E_i = E_1\ or\ E_2$$

We then make our effective modulus dimensionless:

$$\frac{E_{eff}}{E_1} = \frac{1}{\sum_i \frac{E_1}{E_i} * \frac{h_i}{h}}$$



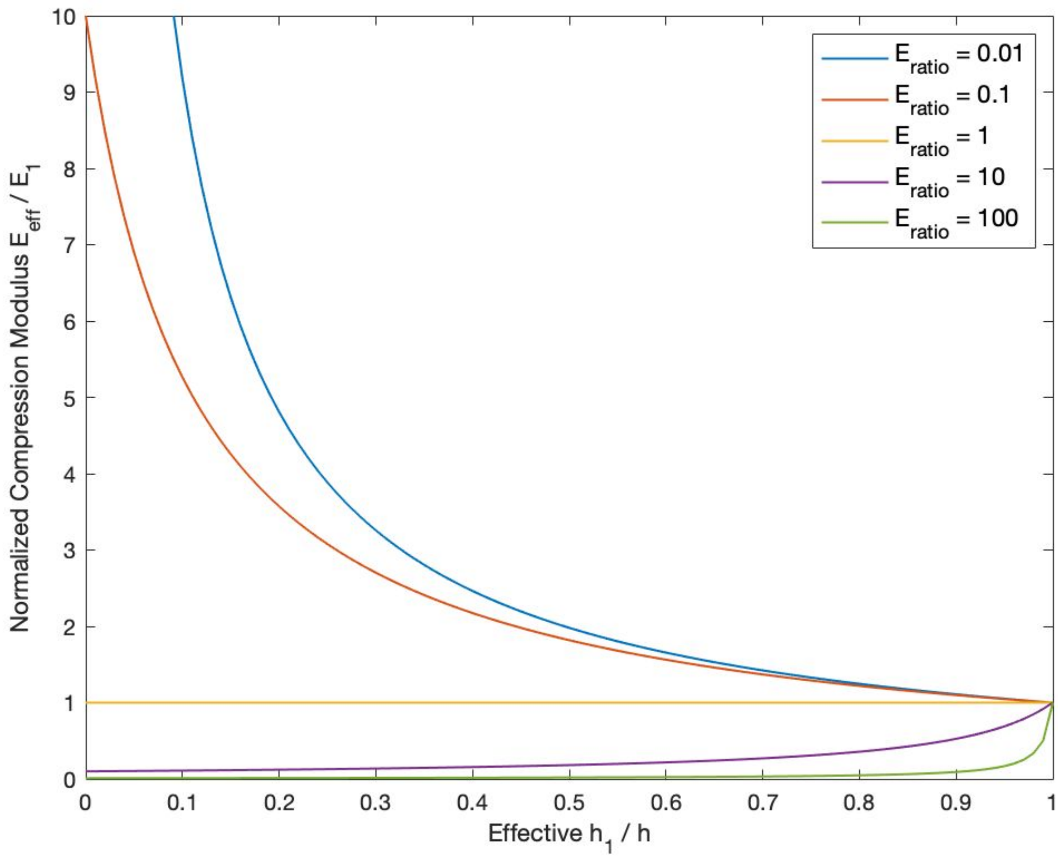
Then, using the above model we can say:

$$\frac{E_{eff}}{E_1} = \frac{1}{\frac{E_1}{E_1} * \frac{h_1}{h} + \frac{E_1}{E_2} * \frac{h_2}{h}}$$
$$where\ h_1\ is\ the\ sum\ of\ all\ material\ 1\ thicknesses$$
$$\&\ h_2\ is\ the\ sum\ of\ all\ material\ 2\ thicknesses$$

And finally:

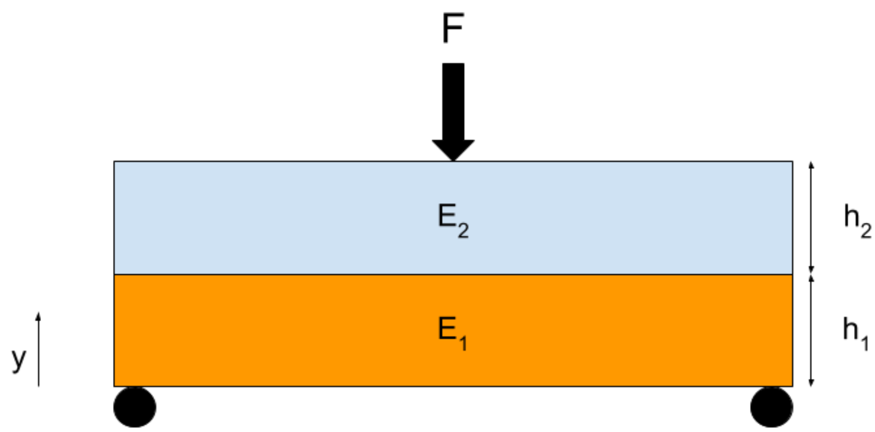
$$\frac{E_{eff}}{E_1} = \frac{1}{\frac{h_1}{h} + E_R * (1 - \frac{h_1}{h})}$$
$$where\ E_R = \frac{E_1}{E_2}$$

Plotting the above expression in Matlab with different values for E_R we get the following figure:



What can be seen from this graph is that when E_R is very low ($E_1 << E_2$), as we increase the ratio of the thickness of material 1 to the overall thickness of the stack the normalized compression modulus (E_{eff} / E_1) starts very high and then decreases to 1. Oppositely, when E_R is very high ($E_1 >> E_2$), as we increase the thickness of material 1 when compared to the overall thickness of the stack the normalized compression modulus (E_{eff} / E_1) slowly increases from 0 to 1.

BENDING



Constitutive:

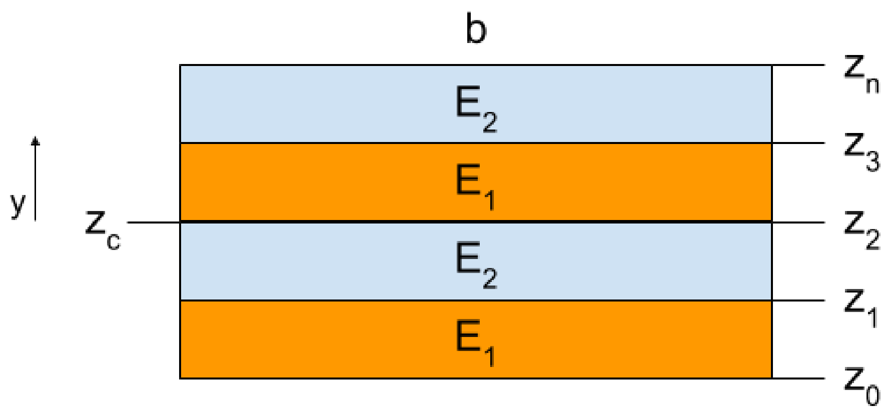
$$E * \varepsilon = \sigma$$

Compatibility:

$$\varepsilon = k * y$$

where y is the distance from the neutral axis
& k is the radius of curvature

Neutral Axis:



We know the stress along the neutral axis is 0:

$$b * \sum_i \int_{A_i} -E_i * k * y * dA = 0 \;\rightarrow\; \sum_i \int_{z_{i-1}}^{z_i} E_i * k * (z - z_c) * dz = 0$$

$$(\tfrac{1}{2}) \sum_i E_i * (z_i^2 - z_{i-1}^2) - \sum_i E_i * z_c * (z_i - z_{i-1}) = 0$$

$$z_c = \frac{\sum_{i=1}^{n} E_i * (z_i^{\,2} - z_{i-1}^{\,2})}{2 * \sum_{i=1}^{n} E_i * (z_i - z_{i-1})}$$

the above equation is used to find the neutral axis relative to the bottom layer of the stack

Equilibrium (Moment Balance):

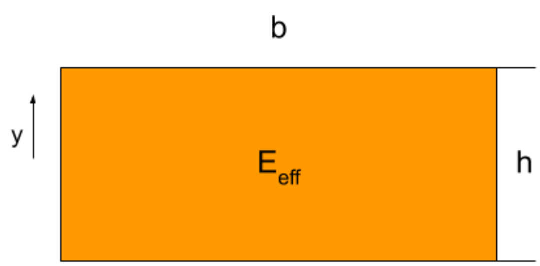
$$\int_{A_i} \sigma * y * dA = \int_{A_i} \sigma * (z - z_c) * d(b * (z - z_c)) + M = 0$$

$$-b * \sum_i \int_{A_i} - E_i * k * (z - z_c) * (z - z_c) * dA + M = 0$$

$$-\sum_i \int_{z_i} E_i * k * (z - z_c) * (z - z_c) * d(b * (z - z_c)) + M = 0$$

$$\sum_i (\tfrac{1}{3}) * E_i * k * (z - z_c)^3 \Big|_{z_{i-1}}^{z_i} * b + M = 0$$

Then comparing to an equivalent one material beam:



$$-\int_y E_{eff} * k * y * y * b * dy + M = 0$$

Equating the two:

$$\sum_{i=1}^{n} (\tfrac{1}{3}) * E_i * (z - z_c)^3 \Big|_{z_{i-1}}^{z_i} = (\tfrac{1}{3}) * E_{eff} * y^3 \Big|_{-h/2}^{h/2}$$

$$\sum_{i=1}^{n} (\tfrac{1}{3}) * E_i * [(z_i - z_c)^3 - (z_{i-1} - z_c)^3] = (\tfrac{1}{12}) * E_{eff} * h^3$$

we then set $I_i = (\tfrac{1}{3}) * [(z_i - z_c)^3 - (z_{i-1} - z_c)^3]$, *leaving us with*

$$\sum_{i=1}^{n} E_i * I_i = (\tfrac{1}{12}) * E_{eff} * h^3$$

Finally, we must remove dimensions:

$$\frac{z_c}{z_n} = \frac{\sum_{i=1}^{n} \frac{E_i}{E_1} * \left(\frac{z_i}{z_n}^2 - \frac{z_{i-1}}{z_n}^2\right)}{2 * \sum_{i=1}^{n} \frac{E_i}{E_1} * \left(\frac{z_i}{z_n} - \frac{z_{i-1}}{z_n}\right)}$$

$$\frac{E_{eff}}{E_1} = 4 * \sum_{i=1}^{n} \frac{E_i}{E_1} * \left[\left(\frac{z_i}{z_n} - \frac{z_c}{z_n}\right)^3 - \left(\frac{z_{i-1}}{z_n} - \frac{z_c}{z_n}\right)^3\right]$$

Using the above expressions I was able to generate some MATLAB code to determine the effective modulus given a stack thickness, a number of layers, and a ratio of moduli, but did not have time to create any plots or visualizations.

MATLAB SCRIPT

```
%% COMPRESSION

clc
clear
close all

% This script is for the calculation of effective compression modulus E (dimensionless)

% Step 1: definition of variables
h1_eff = 0:0.01:1;                          % overall thickness of material 1
h1_eff = h1_eff';
[row, column] = size(h1_eff);
h2_eff = ones([row, column]) - h1_eff;              % overall thickness of material 2

E_ratio = [0.01; 0.1; 1; 10; 100];               % modulus ratio of materials 1/2
B_bar = zeros(row, length(E_ratio));

% Step 2: calculation of effective compression modulus E
for i = 1:5                                 % index for E_ratio

   B_bar(:,i) = 1./(h1_eff + E_ratio(i)*h2_eff);      % dimensionless effective compression modulus B

end

% Step 3: plot results
% Plot the influence of h1_eff on B_bar
figure(1)
for i = 1:5
   plot(h1_eff, B_bar(:,i), 'LineWidth',1);
   hold on

end
hold off

% Figure setup
xlim([0,1]);
ylim([0,10]);
xlabel('Effective h_{1} / h','FontSize',11);
ylabel('Normalized Compression Modulus E_{eff} / E_{1}','FontSize',11);
```

```
legend({'E_{ratio} = 0.01','E_{ratio} = 0.1','E_{ratio} = 1','E_{ratio} = 10','E_{ratio} = 100'},'FontSize',11);

%% BENDING
clc
clear
close all

% This script is for the calculation of effective bending modulus E (dimensionless)

z_n = 1;                              % total stack thickness
n = 4;                                % number of layers
z_vec = 0:(z_n/n):z_n;                % z_1, z_2, z_3...
z_num = zeros(n);
z_den = zeros(n);
E_ratio = [0.01; 0.1; 1; 10; 100]

for i = [2:length(z_vec)]
  if mod(i,2) == 0 % for even items
    z_num(i) = E_ratio*((z_vec(i)^2/z_n^2)-(z_vec(i-1)^2/z_n^2));
    z_den(i) = E_ratio*((z_vec(i)/z_n)-(z_vec(i-1)/z_n));
  else % for odd items
    z_num(i) = ((z_vec(i)^2/z_n^2)-(z_vec(i-1)^2/z_n^2));
    z_den(i) = ((z_vec(i)/z_n)-(z_vec(i-1)/z_n));
  end
end
z_num
z_den
zc = sum(z_num)/(2*sum(z_den))


E = zeros(n,1);

for i = [2:length(z_vec)]
  if mod(i,2) == 0 % for even items
    E(i) = E_ratio*(((z_vec(i)/z_n)-(zc/z_n))^3-((z_vec(i-1)/z_n)-(zc/z_n))^3);
  else % for odd items
    E(i) = (((z_vec(i)/z_n)-(zc/z_n))^3-((z_vec(i-1)/z_n)-(zc/z_n))^3);
  end
end
E

E_eff = 4*sum(E)
```

RESOURCES

1. http://www.aapsm.org/runshoe-running-anatomy.html
2. https://gearpatrol.com/2020/01/08/carbon-running-shoe-buyers-guide/
3. https://www.forbes.com/sites/timnewcomb/2018/11/05/the-technologies-that-define-sneaker-cushioning/#354c1636356e
4. https://onlinelibrary.wiley.com/doi/epdf/10.1002/app.1969.070131106
5. https://www.thieme-connect.com/products/ejournals/html/10.1055/s-2000-7867
6. https://theconversation.com/running-shoes-how-science-can-help-you-to-run-faster-and-more-efficiently-127634