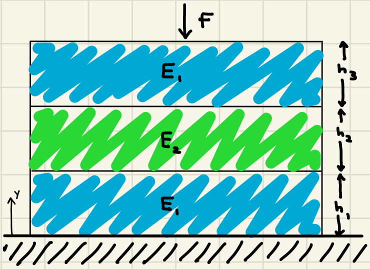
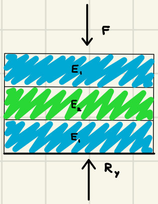
Compression

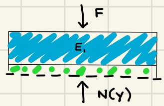


Equilibrium



$\Sigma F_y = 0 = R_y - F$

$R_y = F$



$\Sigma F_y = 0 = N(y) - F$

$N(y) = F$

Compatibility

$$\varepsilon_{1yy} = \frac{\delta_1}{h_1}$$

$$\varepsilon_{2yy} = \frac{\delta_2}{h_2}$$

$$\varepsilon_{nyy} = \frac{\delta_n}{h_n}$$

$$\delta = \delta_1 + \delta_2 + \ldots + \delta_n$$

Constitutive

$$\sigma_{yy} = E_1 \varepsilon_{yy1}$$

$$\sigma_{yy} = E_2 \varepsilon_{yy2}$$

$$N(y) = A\sigma_{yy}$$

$$\sigma_{yy} = \frac{F}{A}$$

assuming constant A

$$AE_1\varepsilon_{1yy} = F = AE_1 \frac{\delta_1}{h_1} \Rightarrow \delta_1 = \frac{Fh_1}{AE_1}$$

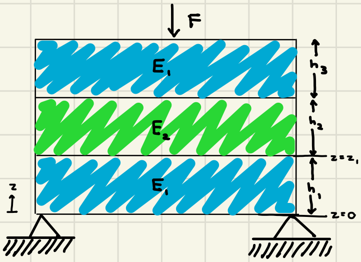
$$\delta = \delta_1 + \delta_2 + \ldots + \delta_n \Rightarrow \delta = \frac{F}{A}\sum_{i=1}^{N} \frac{h_i}{E_i}$$

$$\sigma = E_{eff}\,\varepsilon \Rightarrow \underbrace{\frac{F}{A}}_{\sigma_{eff}} = E_{eff} \underbrace{\frac{\delta}{\Sigma h_i}}_{\varepsilon_{eff}}$$

$$\frac{\Sigma h_i}{\Sigma\, h_i/E_i} = E_{eff}$$

$$\boxed{E_{eff} = \frac{E_1 E_2 \sum_{i=1}^{N} h_i}{\sum_{i=1,3,\ldots}^{N} h_i E_2 + \sum_{i=2,4,\ldots}^{N} h_j E_1}}$$

Bending



Constitutive

$$\sigma_i = E_i \varepsilon_i$$

Compatibility

$$\varepsilon_i = \kappa y = \frac{1}{\rho} y$$

↳ distance from neutral axis

(+) strain on outside (-y)

Finding the neutral axis

Stress along neutral axis = 0

$z_c \rightarrow$ neutral axis

$$\int \sigma_{xx}\, dA = 0 = F$$

$$\int_{A_i} -E_i \kappa (z_i - z_c)\, dA = 0 = \int_{z_{i-1}}^{z_i} E_i (z - z_c)\, dz = 0$$

$$\Rightarrow \sum_i \frac{1}{2} E_i (z_i^2 - z_{i-1}^2) - \sum_i E_i z_c (z_i - z_{i-1}) = 0$$

$$\Rightarrow z_c = \frac{\sum_{i=1}^{N} E_i (z_i^2 - z_{i-1}^2)}{2 \sum_{i=1}^{N} E_i (z_i - z_{i-1})}$$

position of neutral axis from bottom layer z = 0

Moment Balance

for each cross section

$$\int_{A_i} \sigma y A = \sum_i \int_{A_i} \sigma (z - z_c)\, dA + M = 0$$

$$= \sum_i \int_{A_i} \sigma (z - z_c)\, d(b(z - z_c)) \quad + M = 0$$

↳ width of cross section assumed uniform

as one beam with one Eeff

$$-\int_y E_{eff}\, \kappa y y b\, dy + M = 0$$

$$\frac{1}{12} E_{eff} h^3 = M$$

$$-M = \sum_i \int_{A_i} \sigma (z - z_c)\, d(b(z - z_c))$$

$$\cancel{-}M = \cancel{-}b \sum_i \int_{A_i} -E_i \kappa (z - z_c)(z - z_c)\, dA$$

set the two equal

$$M = \sum_i \frac{1}{3} E_i (z - z_c)^3 \Big|_{z_{i-1}}^{z} b \kappa$$

$$M = E_{eff} I \kappa$$

$$\sum_i \frac{1}{3} E_i \underbrace{\left[(z - z_c)^2 - (z_{i-1} - z_c)^3 \right]}_{I} = \frac{1}{12} E_{eff} h^3$$

$$\frac{1}{12} h^3 E_{eff} = \sum_{i=1}^{n} E_i - I_i$$

Need dimensionless

divide by h^3 and z_n

↳ total thickness

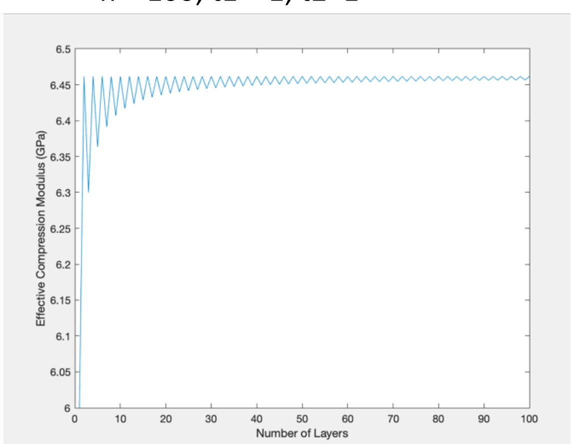
$$\frac{z_c}{z_n} = \frac{\sum_{i=1}^{n} E_i \left(\frac{z_i^2}{z_n} - \frac{z_{i-1}^2}{z_n} \right)}{2 \sum_{i=1}^{n} E_i \left(\frac{z_i}{z_n} - \frac{z_{i-1}}{z_n} \right)}$$

Plugging in you get

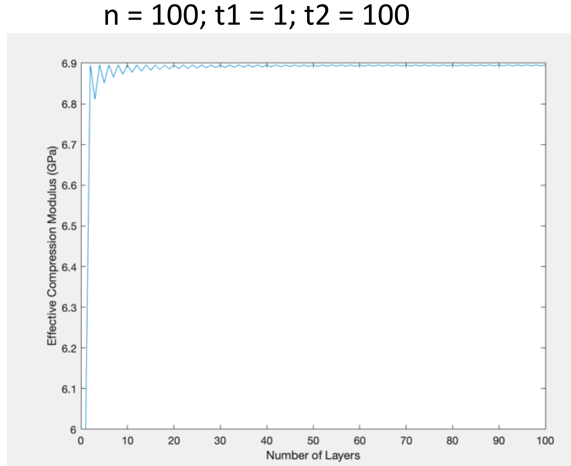
$$\boxed{E_{eff} = 4 \sum_{i=1}^{n} E_i \left[\left(\frac{z_i}{z_n} - \frac{z_c}{z_n} \right)^3 - \left(\frac{z_{i-1}}{z_n} - \frac{z_c}{z_n} \right)^3 \right]}$$

```
% Effective Compression Modulus

% Known individual modulus (in GPa)
E1 = 6;
E2 = 7;
% Given number of layers
n = 10;
% Given thicknesses for n layers
t1 = 1;
t2 = 6;
% Create the layers of given thicknesses
h = ones(n, 1);
for j = 1:length(h)
    if rem(j,2)== 0
        h(j) = h(j)*t2;
    else
        h(j) = h(j)*t1;
    end
end

% The effective modulus for n number of layers is calculated below
Eeff_n = (E1*E2*sum(h))/((sum(h(1:2:end))*E2)+(sum(h(2:2:end))*E1));

% Plot the effective modulus
Eeffs = [E1]; %starts with just one layer with known modulus
for i = 2:n
    Eeffs(i) = (E1*E2*sum(h(1:i))/((sum(h(1:2:i))*E2)+(sum(h(2:2:i))*E1)));
end
plot(1:n,Eeffs);

xlabel('Number of Layers');
ylabel('Effective Compression Modulus (GPa)');
```

n = 100; t1 = 1; t2=1

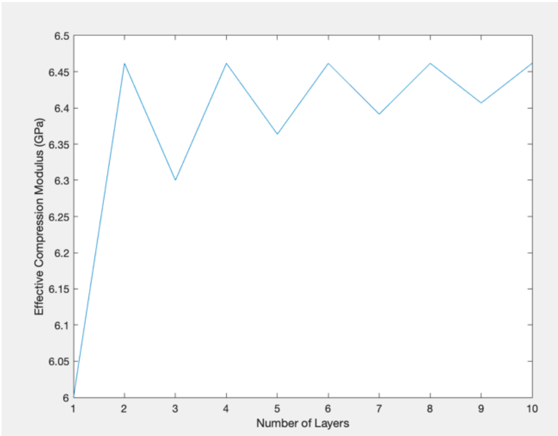


n = 100; t1 = 1; t2 = 100

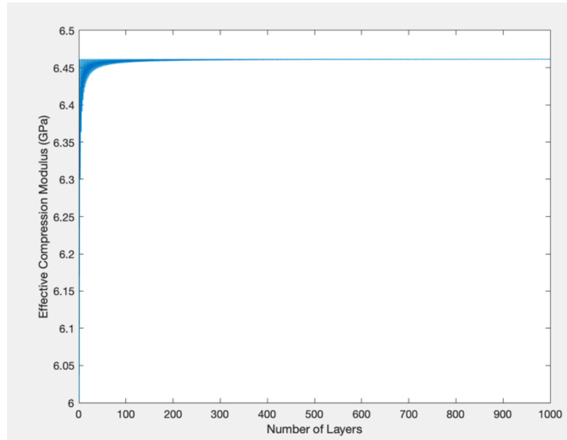


If one material is much thicker than the other, the effective modulus will converge on an effective modulus much faster than if the thicknesses are similar. The effective modulus will also settle at a value closer to the modulus of the thicker material, depending on the ratio between the two thicknesses.

n = 10; t1 = 1; t2=1



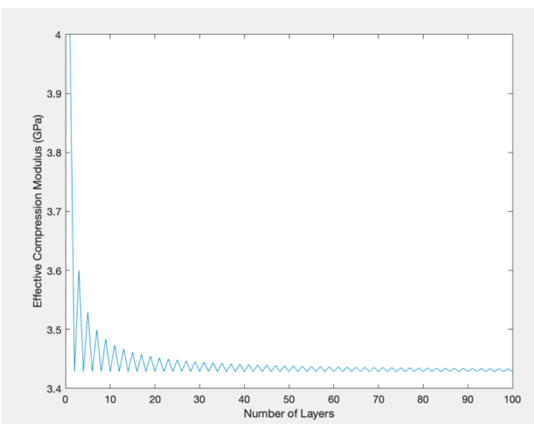
n = 1000; t1 = 1; t2=1



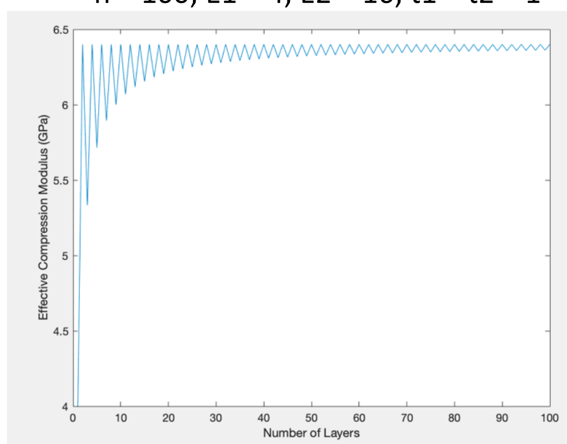
As n increases, the effective modulus begins to converge on a value, while at lower n, it will vary slightly from layer to layer.

The above two cases were done with E1 = 6 and E2 = 7 GPa.

n = 100; E1 = 4; E2 = 3; t1 = t2 = 1

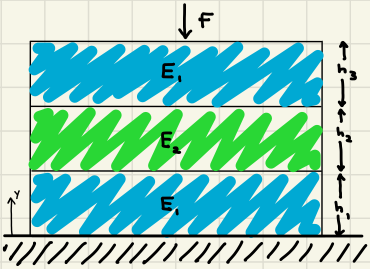


n = 100; E1 = 4; E2 = 16; t1 = t2 = 1

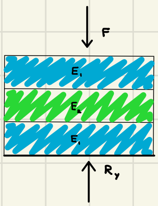


As the ratio between the two individual modulus values increases, the effective modulus will vary more from layer to layer, so the amplitude of the oscillations is larger on the graph.

Compression



Equilibrium



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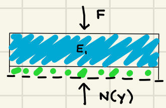

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