ISR Theory 2

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Warning on Notation

Engineering Notation

$$e^{j\omega t - j\mathbf{k}\cdot\mathbf{r}}$$

- Used in most antenna theory textbooks
- Used in the Kudeki and Milla [2011] IEEE review paper on ISR theory

Conversion between notations is $j = -i$

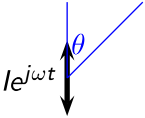
Physicist Notation

$$e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}$$

- Used in Jackson E&M textbook
- Used in most plasma physics textbooks
- Used for solution to Landau problem

Hertzian Dipole Antenna

Consider an infinitesimal dipole antenna of length $d\ell$ carrying current a sinusoidal current I



$$\mathbf{J} = Id\ell\,\delta\left(\mathbf{x}\right)\hat{z}e^{j\omega t}$$

Far Field Solution ($\eta_0 \equiv \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\ \Omega$)

$$\mathbf{E}_{ff} = \frac{jk_0\eta_0 Id\ell}{4\pi r}\sin\theta e^{j\omega t - jk_0 r}\hat{\theta}$$
$$\mathbf{B}_{ff} = \frac{jk_0\mu_0 Id\ell}{4\pi r}\sin\theta e^{j\omega t - jk_0 r}\hat{\phi}$$

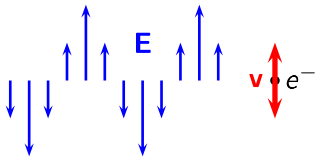
Far Field Radiated Power

$$S = \frac{1}{2\mu_0}\Re\left\{\mathbf{E}\times\mathbf{B}^*\right\} = \frac{1}{2\eta_0}\left|E\right|^2 = \frac{\eta_0}{2}\left(\frac{k_0 Id\ell}{4\pi r}\right)^2\sin^2\theta$$

Thompson Scatter from One Electron

Incident wave:

$$\mathbf{E} = \hat{z} E_0 e^{j\omega t - j\mathbf{k}_0 \cdot \mathbf{r}}$$



Motion of the electron:

$$j\omega m_e \mathbf{v} = -e\mathbf{E} \rightarrow \mathbf{v} = \frac{je}{\omega m_e} E_0 \hat{z}$$

Effective Hertzian Dipole with $Id\ell \rightarrow e\mathbf{v}$ (also note $\omega / k = c$)

$$\mathbf{E}_{scat} = \frac{-\eta_0 e^2}{4\pi r m_e c} E_0 \sin\theta e^{j\omega t - j\mathbf{k}_0 \cdot \mathbf{r}} \hat{\theta} = -\frac{r_e}{r} E_0 e^{j\omega t - j\mathbf{k}_0 r} \hat{\theta}$$

Where the classical electron radius is

$$r_e = \frac{\eta_0 e^2}{4\pi m_e c} = \frac{e^2}{4\pi \epsilon_0 m_e c^2} \approx 2.818 \times 10^{-15} \text{ m}$$

Thompson Scatter Cross section

Total Cross Section:

$$\sigma_t \equiv \frac{P_{\rm tot}}{\frac{1}{2\eta_0}\left|E_0\right|^2} = \frac{8\pi}{3} r_e^2 \qquad \text{Where } P_{\rm tot} \equiv \int_0^{2\pi}\int_0^{\pi} S_{scat}\, r^2 \sin\theta \, d\theta d\phi$$

Radar Cross Section:

$$\sigma = \sigma_t D_s$$

Directivity of scattering in the direction towards the radar:

$$D_s \equiv \frac{S_{scat}(\text{at the radar})}{\frac{P_{\rm tot}}{4\pi r^2}}$$

For a Herztian dipole, $S \propto \frac{\sin\theta}{r^2}$, $D_s(\theta,\phi) = \frac{3}{2}\sin^2\theta$.
For backscatter $\theta = 180^\circ$, so the radar cross section of one electron is

$$\sigma = 4\pi r_e^2 \approx 10^{-28}\ \mathrm{m}^2 \qquad (\sim 0.9979 \times 10^{-28}\mathrm{m}^2)$$

Why Can We Ignore the Ions?

$$\sigma_e \propto \frac{1}{m_e^2}$$

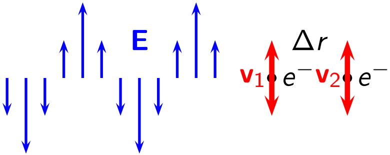
The scattering cross section of an ion is

$$\sigma_i = \frac{m_e^2}{m_i^2}\sigma_e$$

For an O^+ plasma

$$\frac{m_e^2}{m_i^2} = 1.16 \times 10^{-9}$$

Scatter from Two Electrons



Incident on first electron:

$$E_1 = E_0 e^{j\omega t}$$

Scattered from first electron:

$$\begin{aligned} E_{s1} &= -\frac{r_e}{r} E_1 e^{-jk_0 r} \\ &= -\frac{r_e}{r} E_0 e^{j\omega t - jk_0 r} \end{aligned}$$

Incident on second electron:

$$E_2 = E_0 e^{j\omega t - jk_0 \Delta r}$$

Scattered from second electron:

$$\begin{aligned} E_{s2} &= -\frac{r_e}{r + \Delta r} E_2 e^{-jk(r+\Delta r)} \\ &= -\frac{r_e}{r} E_0 e^{j\omega t - jk_0 r - j2k_0 \Delta r} \end{aligned}$$

In the far field $\frac{1}{r+\Delta r} \approx \frac{1}{r}$, so the sum of the fields is

$$E_{s1} + E_{s2} = -\frac{r_e}{r} E_0 e^{j\omega t - jk_0 r} \left(1 + e^{-j2k_0 \Delta r}\right)$$

Bragg Wavelength

For scatter from two electrons

$$|E_{s1} + E_{s2}|^2 \propto \left|1 + e^{-j2k_0\Delta r}\right|^2 = 4\cos^2(k_0\Delta r)$$

- If $\Delta r = \frac{\lambda}{2}$, $k_0\Delta r = \pi$, and the factor is 4 (perfect constructive interference)
- If $\Delta r = \frac{\lambda}{4}$, $k_0\Delta r = \frac{\pi}{2}$, and the factor is 0 (perfect destructive intereference)
- If Δr is a random number, the expected value of the factor is 2.

The Bragg wavelength $\lambda_b = \frac{\lambda_0}{2}$ is the preferred spacing where the scatter adds constructively.

Define the Bragg wavenumber (for backscatter) as $k_b = \frac{2\pi}{\lambda_b} = \frac{4\pi}{\lambda_0} = 2k_0$.

Scatter from Many Electrons

$$E_s = -\frac{r_e}{r} E_0 e^{j\omega t - jkr} \left(\sum_{p=0}^{N-1} e^{-j2\mathbf{k}_0 \cdot \Delta \mathbf{r}_p} \right)$$
$$= -\frac{r_e}{r} E_0 e^{j\omega t - jk_0 r} \int n_e(\mathbf{r}) \, e^{-j2\mathbf{k}_0 \cdot \Delta \mathbf{r}_p} d^3 r$$

where the microscopic electron density is

$$n_e(\mathbf{r}) \equiv \sum_{p=0}^{N-1} \delta\left(\mathbf{r} - \Delta \mathbf{r}_p\right)$$

This looks like a spatial Fourier transform evaluated at the Bragg wavenumber $\mathbf{k}_b = 2\mathbf{k}_0$.

The scatter is most sensitive to density structures at the Bragg wavelength.

Coherent vs Incoherent Scatter

- **Coherent Scatter**: If the plasma is unstable and full of irregularities at the Bragg wavelength, lots of constructive interference will occur, and the radar will receive lots of signal.
- **Incoherent Scatter**: The plasma is disorganized:

$$\left| \sum_{p=0}^{N-1} e^{-j2\mathbf{k}_0 \cdot \Delta \mathbf{r}_p} \right|^2 \approx N$$

The pathological case were no scatter is received due to perfect destructive interference will almost surely never happen with a large number of electrons.

Rough Dectectability Calculations

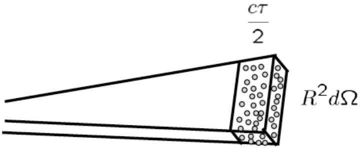
Radar Equation:

$$P_r = P_t \frac{G}{4\pi r^2} \sigma \frac{A_{eff}}{4\pi r^2}$$

For a distribution of electrons:

$$\sigma = \sigma_e N_e V \approx \sigma_e N_e r^2 \frac{c\tau_p}{2} \frac{4\pi}{G}$$

$$P_r \approx P_t \sigma_e N_e \frac{c\tau}{2} \frac{A_{eff}}{4\pi r^2}$$



Approximate beam solid angle:

$$d\Omega \approx \frac{4\pi}{G}$$

For $P_t = 1$ MW, $N_e = 10^{11}\ \mathrm{m}^{-3}$, $\tau = 500\ \mu\mathrm{s}$, $r = 300$ km, $A_{eff} \approx 0.6 A_{geo}$, $A_{geo} = \frac{\pi}{4} d^2$, and a dish diameter of $d = 300$ m, this gives:

$$P_r = 2.81 \times 10^{-14}\ \mathrm{W}$$

For a smaller radar with $d = 30$ m, $P_r = 2.81 \times 10^{-16}$ W

Radio Noise

Nyquist Noise Theorem: $P_N = k_B T_{sys} B$

- A good UHF receiver will have a $T_{sys} \approx 125\ \mathrm{K}$.
- B is the receiver bandwidth.

Doppler shift from electron thermal motion:

$$\Delta f = \frac{2}{c} f_{\mathrm{Tx}} v \approx \frac{2}{c} f_{\mathrm{Tx}} \sqrt{\frac{k_B T_e}{m_e}}$$

Let's assume we need to capture $B = 4\Delta f$ to get the full spectrum.
For $f_{\mathrm{Tx}} = 450\ \mathrm{MHz}$ and $T_e = 1000\ \mathrm{K}$:

$$B = 1.48\ \mathrm{MHz} \Rightarrow P_N = 2.55 \times 10^{-15}\ \mathrm{W}$$

What if instead the bandwidth is related to the ion motion?

$$v_i = \sqrt{\frac{m_e}{m_i}} v_e \Rightarrow v_i = 5.83 \times 10^{-3} v_e \text{ for } \mathrm{O}^+$$

The same numbers would yield

$$B = 8.63\mathrm{kHz} \Rightarrow P_N = 1.48 \times 10^{-17}\ \mathrm{W}$$

Autocorrelation Functions and Power Spectra

What determines the bandwidth of the received signal?

The electrons are moving, such that $\Delta\mathbf{r}_p(t)$ is a function of time.

Autocorrelation function between scatter at two different times, t and $t+\tau$:

$$\langle E_s^*(t)\, E_s(t+\tau)\rangle = \frac{r_e^2}{r^2}|E_0|^2 \left\langle \left[\sum_{p=0}^{N-1} e^{+j2\mathbf{k}_0\cdot\Delta\mathbf{r}_p(t)}\right] \left[\sum_{q=0}^{N-1} e^{-j2\mathbf{k}_0\cdot\Delta\mathbf{r}_q(t+\tau)}\right] \right\rangle$$

$$= \frac{r_e^2}{r^2}|E_0|^2 \left\langle \sum_{p=0}^{N-1}\sum_{q=0}^{N-1} e^{j2\mathbf{k}_0\cdot(\Delta\mathbf{r}_p(t)-\Delta\mathbf{r}_q(t+\tau))} \right\rangle$$

Power spectrum of scatter:

$$\left\langle |E_s(\omega)|^2 \right\rangle \equiv \int_{-\infty}^{\infty} \langle E_s^*(t)\, E_s(t+\tau)\rangle\, e^{-j\omega\tau}\, d\tau$$

$$= \frac{r_e^2}{r^2}|E_0|^2 \left\langle |n_e(\mathbf{k}_b,\omega)|^2 \right\rangle$$

Simplest Possible Case

- The particles are non interacting
  - $\left\langle e^{j2\mathbf{k}_0\cdot(\Delta\mathbf{r}_p(t)-\Delta\mathbf{r}_q(t+\tau))}\right\rangle = 0$ for $p \neq q$.
- The particles move in straight lines at constant velocities $\mathbf{v}_p$ (positive away from radar)
  - $\Delta\mathbf{r}_p(t) = \Delta\mathbf{r}_p(0) + \mathbf{v}_p t$

$$\begin{aligned}
\langle E_s^*(t)\,E_s(t+\tau)\rangle &= \frac{r_e^2}{r^2}|E_0|^2\left\langle \sum_{p=0}^{N-1}\sum_{q=0}^{N-1} e^{j2\mathbf{k}_0\cdot(\Delta\mathbf{r}_p(t)-\Delta\mathbf{r}_q(t+\tau))}\right\rangle \\
&= \frac{r_e^2}{r^2}|E_0|^2\left\langle \sum_{p=0}^{N-1} e^{j2\mathbf{k}_0\cdot(\Delta\mathbf{r}_p(t)-\Delta\mathbf{r}_p(t+\tau))}\right\rangle \\
&= \frac{r_e^2}{r^2}|E_0|^2\left\langle \sum_{p=0}^{N-1} e^{j2\mathbf{k}_0\cdot(-\mathbf{v}_p\tau)}\right\rangle \\
&= \frac{r_e^2}{r^2}|E_0|^2\int f(\mathbf{v})\,e^{-j2\mathbf{k}_0\cdot\mathbf{v}\tau}\,d^3v
\end{aligned}$$

Doppler Spectrum in Simplest Possible Case

$$\left\langle \left|E_s\left(\omega\right)\right|^2 \right\rangle \equiv \int_{-\infty}^{\infty} \left\langle E_s^*\left(t\right) E_s\left(t+\tau\right)\right\rangle e^{-j\omega\tau}\, d\tau$$

$$= \frac{r_e^2}{r^2} \left|E_0\right|^2 \int_{-\infty}^{\infty} d\tau e^{-j\omega\tau} \int d^3 v f\left(\mathbf{v}\right) e^{-j2\mathbf{k}_0\cdot\mathbf{v}\tau}$$

$$= \frac{r_e^2}{r^2} \left|E_0\right|^2 \int d^3 v f\left(\mathbf{v}\right) \delta\left(-\omega - 2\mathbf{k}_0\cdot\mathbf{v}\right)$$

$$= \frac{r_e^2}{r^2} \left|E_0\right|^2 f\left(-\frac{\omega}{2k_0}\hat{k}\right)$$

A particle moving at velocity $\mathbf{v}$ backscatters with a Doppler shift of $\Delta\omega = -2\mathbf{k}_0 \cdot \mathbf{v}$.

Spectrum of received frequencies is directly related to spectrum of particle velocities.

Particle Trajectories in Plasmas

Factors complicating particle trajectories

- Background magnetic field
- Collisions with neutrals
- Coulomb collisions with other particles
- Forces from the self-consistent fields generated by all other particles

The first three effects can be treated by deriving more complicated $\Delta \mathbf{r}_p(t)$ for each particle.

The collective effects, however, are much more complicated to treat.

Debye Length

At what scale are collective effects important?

- Characteristic particle velocity: Electron thermal speed

$$v_{te} = \sqrt{\frac{k_B T_e}{m_e}}$$

- Characteristic time scale for collective interacttions: Inverse electron plasma frequency

$$\tau_e = \frac{1}{\omega_{pe}} = \sqrt{\frac{m_e \epsilon_0}{e^2 N_e}}$$

- Characteristic length scale: Debye length

$$\lambda_{De} = v_{te}\tau_e = \sqrt{\frac{\epsilon_0 k_B T_e}{e^2 N_e}}$$

Collective effects will have a significant affect on the particle trajectories over a Bragg wavelength if $\lambda_b > \lambda_{De}$

Particle-in-Cell Simulations

Electrostatic PIC equations:

$$m_s \frac{dv_s}{dt} = q_s \mathbf{E}$$

$$\mathbf{E} = -\nabla \Phi$$

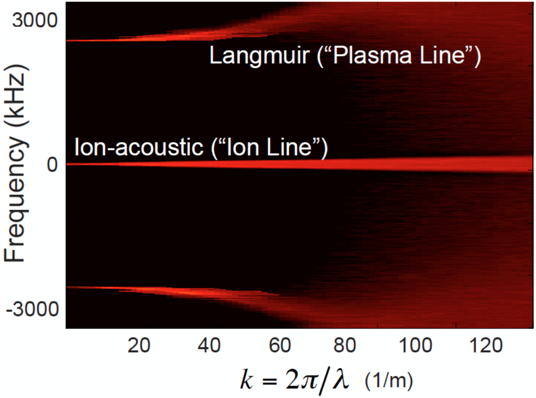
$$\nabla^2 \Phi = -\frac{\rho_c}{\epsilon_0}$$

Diaz et al., RS [2008].

An ISR would pick out one slice of this spectrum at $\mathbf{k} = \mathbf{k}_b$.

Computed density fluctuation spectrum

$$\left\langle \left| n_e \left(\mathbf{k}, \omega \right) \right|^2 \right\rangle$$



Dressed Particle Theory

- Imagine a set of "test particles" that move without experiencing collective effects.
- As the charged test particles move they create macroscopic disturbances in the rest of the plasma.

$$\left\langle \left| n_e\left(\mathbf{k},\omega\right)\right|^2 \right\rangle \approx \left\langle \left| n_{te}\left(\mathbf{k},\omega\right) + \delta N_e\left(\mathbf{k},\omega\right)\right|^2 \right\rangle$$

Where

- n_{te} is the microscopic density function of the test electrons
- δN_e are the macroscopic density fluctuations in response to the test particles (ions and electrons)

Plasma Response Functions

Plasma as a generalized conductor:

$$\mathbf{J}_e = \sigma_e(\mathbf{k}, \omega)\,\mathbf{E}$$

Plasma as a generalized dielectric:

$$\mathbf{P}_e = \epsilon_0 \chi_e(\mathbf{k}, \omega)\,\mathbf{E}$$

Bound charge density and electron density disturbance

$$\rho_c = -\nabla \cdot \mathbf{P} \rightarrow \delta N_e(\mathbf{k}, \omega) = +\frac{\epsilon_0}{e}\nabla \cdot \left[\chi_e(\mathbf{k}, \omega)\,\mathbf{E}\right]$$

These are equivalent descriptions

$$\mathbf{J}_e = \frac{\partial}{\partial t}\mathbf{P}_e \rightarrow \chi_e(\mathbf{k}, \omega) = \frac{\sigma_e(\mathbf{k}, \omega)}{j\omega\epsilon_0}$$

Gauss' Law

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho_c$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = -\nabla \cdot (\mathbf{P}_i + \mathbf{P}_e) + en_{ti} - en_{te}$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = -\epsilon_0 \nabla \cdot [(\chi_i + \chi_e) \mathbf{E}] + en_{ti} - en_{te}$$

$$\epsilon_0 \nabla \cdot [(1 + \chi_i + \chi_e) \mathbf{E}] = en_{ti} - en_{te}$$

$$-j\epsilon_0 (1 + \chi_i + \chi_e) \mathbf{k} \cdot \mathbf{E} = en_{ti} - en_{te}$$

$$\mathbf{k} \cdot \mathbf{E} = \frac{en_{ti} - en_{te}}{-j\epsilon_0 (1 + \chi_i + \chi_e)}$$

Dielectric permittivity of the plasma:

$$\epsilon = \epsilon_0 (1 + \chi_i + \chi_e)$$

Effective electron density fluctuations

$$
\begin{aligned}
n_e &= n_{te} + \delta N_e \\
&= n_{te} + \frac{\epsilon_0}{e} \nabla \cdot [\chi_e \mathbf{E}] \\
&= n_{te} - j \frac{\epsilon_0}{e} \chi_e \mathbf{k} \cdot \mathbf{E} \\
&= \frac{1 + \chi_i}{1 + \chi_i + \chi_e} n_{te} + \frac{\chi_e}{1 + \chi_i + \chi_e} n_{ti}
\end{aligned}
$$

Power spectrum (note n_{te} and n_{ti} are uncorrelated)

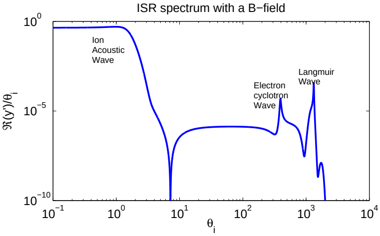
$$
\begin{aligned}
\left\langle \left| n_e (\mathbf{k}, \omega) \right|^2 \right\rangle = & \frac{\left|1 + \chi_i\right|^2}{\left|1 + \chi_i + \chi_e\right|^2} \left\langle \left| n_{te} (\mathbf{k}, \omega) \right|^2 \right\rangle \\
& + \frac{\left|\chi_e\right|^2}{\left|1 + \chi_i + \chi_e\right|^2} \left\langle \left| n_{ti} (\mathbf{k}, \omega) \right|^2 \right\rangle
\end{aligned}
$$

Connection to Dispersion Relation for Electrostatic Waves

- If there were no test particle to drive fluctuations, Gauss' Law of the macroscopic plasma is

$$\epsilon_0 \nabla \cdot \left[\left(1 + \chi_i + \chi_e \right) \mathbf{E} \right] = 0$$

- If $1 + \chi_i (\mathbf{k}, \omega) + \chi_e (\mathbf{k}, \omega) = 0$, then **E** could be anything.
- The set of **k** and ω that satisfy $1 + \chi_i (\mathbf{k}, \omega) + \chi_e (\mathbf{k}, \omega) = 0$ define the **normal modes** in the plasma.
- Near a normal mode, the denominator of the ISR spectrum is nearly 0, so the spectrum has a peak.



Filtered Noise

Can you hear the ocean in a seashell?

Resonant Filter

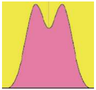
Ambient Noise ⟶

⟶ Colored Noise (Ocean Sounds)

Analogy to ISR:

Thermal Motion ⟶ Collective Effects ⟶ ISR Spectrum

$\left\langle \left| n_{ti}\left(\mathbf{k},\omega\right)\right|^2 \right\rangle$ $\qquad \frac{|\chi_e|^2}{|1+\chi_i+\chi_e|^2}$



Linearized Vlasov Equations

Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}}$$

Linearization assumptions: $f \approx f_0 + f_1 e^{j\omega t - j\mathbf{k}\cdot\mathbf{r}}$

$$j\omega f_1 - j\mathbf{v}\cdot\mathbf{k} f_1 + \frac{q}{m}\frac{\partial f_0}{\partial \mathbf{v}} \cdot \mathbf{E} = 0$$

$$f_1 = -\frac{q}{m}\frac{\frac{\partial f_0}{\partial \mathbf{v}}}{j\left(\omega - \mathbf{k}\cdot\mathbf{v}\right)} \cdot \mathbf{E}$$

Electron density perturbation:

$$\delta N = \int f_1 d^3 v = -\frac{q}{m}\int d^3 v \frac{\frac{\partial f_0}{\partial \mathbf{v}}}{j\left(\omega - \mathbf{k}\cdot\mathbf{v}\right)} \cdot \mathbf{E}$$

$$= j\frac{\epsilon_0}{q}\chi \mathbf{k} \cdot \mathbf{E}$$

Landau Problem

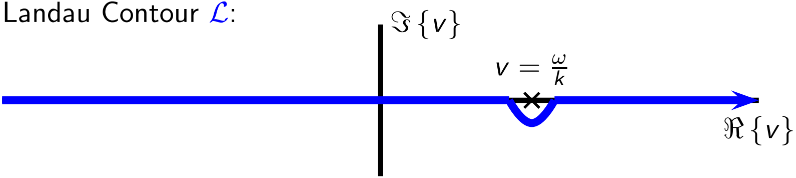
Susceptibilty integral has singularity where $\omega = \mathbf{k} \cdot v$

$$\chi\left(\mathbf{k},\omega\right) = \frac{q^2}{k^2 m\epsilon_0} \int d^3v \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}}$$

Solution: Treat the problem with Laplace transforms (Landau [1946])

$$\begin{aligned}\chi\left(\mathbf{k},\omega\right) &= \frac{q^2}{k^2 m\epsilon_0} \int_{\mathcal{L}} d^3v \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}} \\ &= \frac{q^2}{k^2 m\epsilon_0} \left\{ \mathcal{P} \int d^3v \frac{\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}} + i\pi\, \mathbf{k} \cdot \left.\frac{\partial f_0}{\partial \mathbf{v}}\right|_{\mathbf{v}=\frac{\omega}{k}\hat{k}} \right\}\end{aligned}$$

Landau Contour $\mathcal{L}$:



Relationship of Spectrum to Density and Temperature

Everything is a function of the zeroth-order distribution functions

- Susceptibilities:

$$\chi(\mathbf{k},\omega) = \frac{q^2}{k^2 m\epsilon_0}\int_{\mathcal{L}} d^3v \frac{\mathbf{k}\cdot\frac{\partial f_0}{\partial \mathbf{v}}}{(\omega - \mathbf{k}\cdot\mathbf{v})}$$

- Test particle spectra:

$$\left\langle |n_t(\mathbf{k},\omega)|^2 \right\rangle = f_0\left(-\frac{\omega}{k}\hat{k}\right)$$

If you assume f_0 is Maxwellian, then f_0 is a function of density, bulk velocity, and temperature. Thus the whole ISR spectrum can be written a function of N_e, T_e, T_i, and $\mathbf{u}$.

Total Scattered Power

$$\left\langle \left| n_e(\mathbf{k},\omega) \right|^2 \right\rangle = \frac{\left|1+\chi_i\right|^2}{\left|1+\chi_i+\chi_e\right|^2} \left\langle \left| n_{te}(\mathbf{k},\omega) \right|^2 \right\rangle \qquad \text{Electron Line}$$

$$+ \frac{\left|\chi_e\right|^2}{\left|1+\chi_i+\chi_e\right|^2} \left\langle \left| n_{ti}(\mathbf{k},\omega) \right|^2 \right\rangle \qquad \text{Ion Line}$$

Area under spectrum

$$\int \left\langle \left| n_e(\mathbf{k},\omega) \right|^2 \right\rangle \frac{d\omega}{2\pi} = \frac{k^2\lambda_{De}^2 N_e}{1+k^2\lambda_{De}^2} + \frac{N_e}{\left(1+k^2\lambda_{De}^2\right)\left(1+k^2\lambda_{De}^2+\frac{T_e}{T_i}\right)}$$

In the limit $k^2\lambda_{De}^2 \gg 1$, electron line dominates (wide bandwidth)

$$\int \left\langle \left| n_e(\mathbf{k},\omega) \right|^2 \right\rangle \frac{d\omega}{2\pi} = N_e$$

In the limit $k^2\lambda_{De}^2 \ll 1$, the ion line dominates (narrow bandwidth)

$$\int \left\langle \left| n_e(\mathbf{k},\omega) \right|^2 \right\rangle \frac{d\omega}{2\pi} = \frac{N_e}{1+\frac{T_e}{T_i}}$$

Big Takeaways

- Radars only scatter from electrons
- Bragg scattering effectively picks out a single wavenumber
- Even in a homogeneous, stable plasma, the scatter is never zero
- Ions encode information into the spectrum through collective effects
- ISR theory is intimately connected to dispersion relationships for electrostatic plasma waves
- Plasma normal modes $\Rightarrow$ peaks in spectrum