RANDOM MATRIX THEORY.

16/7/19.

- $\boxed{\text{RMT} = \text{Linear algebra} + \text{Probability theory.}}$
- ~~Energy~~ Lines in nuclear spectra = spacing b/w eigenvalues.
- If we have an $n \times n$ matrix

exp(−s)
uncorrelated hamiltonian
H = diag
H = non-diag
W.S

joint PDF of all elements

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \\ M_{n1} & \cdots & M_{nn} \end{pmatrix} \quad \& \quad P(M_{11}, \dots, M_{nn})$$

What is the distribution of eigenvalues?

- 2×2 random matrices : spacing distribution

consider $x = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$ $\quad x_1, x_2 \sim N(0,1)$, $\quad x_3 \sim N(0, 1/2)$ called the gaussian unitary ensemble.

There is a reason why x_3 has variance $1/2$. We will come back to it later.

What is the PDF of the spacing $s = \lambda_2 - \lambda_1$?

We can get the eigenvalues λ_1 & λ_2 as

$$\lambda_1, \lambda_2 = \frac{-(x_1+x_2) \pm \sqrt{(\underbrace{x_1+x_2}_{\text{trace}(x)})^2 - 4(\underbrace{x_1x_2 - x_3^2}_{\det(x)})}}{2}$$

$$\Rightarrow \quad s = \lambda_2 - \lambda_1 = \sqrt{(x_1+x_2)^2 - 4(x_1x_2 - x_3^2)}$$

$$= \sqrt{(x_1-x_2)^2 + 4x_3^2}$$

Now, we know the PDF of x_1, x_2, x_3. What is the PDF of the r.v s? This should be computable from the PDF's of x_1, x_2, x_3.

now, general trick:

If we know the joint PDF of a bunch of r.v's $x_1, x_2, \ldots x_n$.

and we want to calculate the PDF of $f(x_1, x_2, \ldots x_n)$

$$\boxed{PDF(f=s) = \int_{-\infty}^{\infty} PDF(x_1, \ldots x_n) \; \delta(f(x_1, \ldots x_n) - s) \, dx_1 \ldots dx_n}$$

probability that the function takes a value s.

NOTE: The δ function picks out the value of the joint probability density at each point.

Coming back to original problem

$$P(s) = \int_{-\infty}^{\infty} \frac{dx_1}{\sqrt{2\pi}} \frac{dx_2}{\sqrt{2\pi}} \frac{dx_3}{\sqrt{\pi}} \exp\Big[\overbrace{-\frac{1}{2}(x_1^2+x_2^2) - x_3^2}^{\text{normally distributed r.v's}}\Big]$$

$$\delta\Big(s - \underbrace{\sqrt{(x_1-x_2)^2 + 4x_3^2}}_{f(x_1, x_2, x_3)}\Big)$$

since we are dealing with squares of r.v's, polar coordinates is a natural choice to do the integration

$$x_1 - x_2 = r\cos\theta \qquad x_1 + x_2 = \psi$$
$$x_3 = r\sin\theta$$

$$\Rightarrow \quad x_1 = \frac{\psi + r\cos\theta}{2}, \qquad x_2 = \frac{\psi - r\cos\theta}{2}$$

$$x_3 = r\sin\theta$$

The Jacobian for this variable transformation is given by:

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial \psi} \\ \vdots & & \\ \frac{\partial x_3}{\partial r} & \frac{\partial x_3}{\partial \theta} & \frac{\partial x_3}{\partial \psi} \end{pmatrix} = \begin{pmatrix} \frac{\cos\theta}{2} & -\frac{r\sin\theta}{2} & \frac{1}{2} \\ -\frac{\cos\theta}{2} & \frac{r\sin\theta}{2} & \frac{1}{2} \\ \frac{\sin\theta}{2} & \frac{r\cos\theta}{2} & 0 \end{pmatrix}$$

$$\det J = -\frac{r}{4}$$

- another general trick

When you want to do a variable transform in integration from $x_1, x_2, \dots x_n$ to $\eta_1, \eta_2, \dots \eta_n$

$$\int_{dom(x_1 \dots x_n)} f(x_1, x_2, \dots x_n)\, dx_1 \dots dx_n$$

$$= \int_{dom(\eta_1 \dots \eta_n)} \underbrace{|\det(J)|}_{\text{this will depend on } \eta_1 \dots \eta_n} f[x_1(\eta_1 \dots \eta_n), x_2(\eta_1 \dots \eta_n), \dots x_n(\eta_1 \dots \eta_n)]\, d\eta_1 \dots d\eta_n$$

- use the $x = r\cos\theta$, $y = r\sin\theta$ example
- point out how doing this preserves the lebesgue measure
- eigenvalues capture stretching

- Using above trick, the integral

$$PDF(f = s) = \int_{-\infty}^{\infty}\int \frac{r}{4} \frac{1}{\sqrt{4\pi^3}} \delta(r-s) \cdot \exp\left[-\left(\frac{r^2}{4} + \frac{\psi^2}{4}\right)\right] dr\, d\psi \int_0^{2\pi} d\theta$$

using δ-function

$$= \int_{-\infty}^{\infty} \frac{s\,\theta(s)}{4} \frac{1}{\sqrt{4\pi^3}} \exp\left[-\left(\frac{s^2+\psi^2}{4}\right)\right] d\psi \int_0^{2\pi} d\theta$$

Heaviside step function. This comes in because integration limits of r is from 0 to ∞

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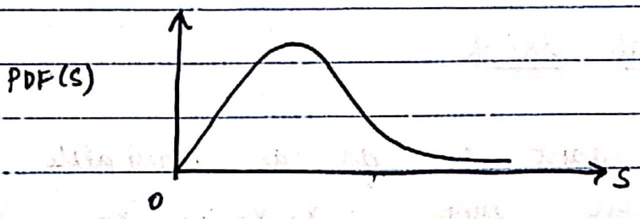
$$\Rightarrow \text{PDF}(f=s) = \frac{1}{4\sqrt{4\pi^3}} \, s\,\theta(s)\,\exp\left[\frac{-s^2}{2}\right] \cdot 2\pi \int_{-\infty}^{\infty} \underbrace{\exp\left[-\frac{\psi^2}{2}\right] d\psi}$$

↳ simple gaussian integral $\approx \sqrt{4\pi}$

$$\Rightarrow \boxed{\text{PDF}(f=s) = \frac{s}{2}\,\theta(s)\,e^{-s^2/4}}$$

Wigner's surmise



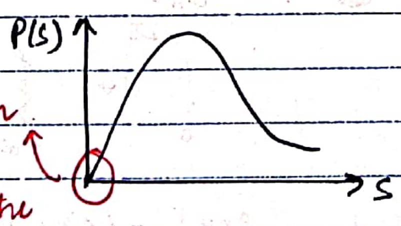
- Think about:

  - What if the matrix elements x_1, x_2, x_3 had a non-zero mean or different values of variances? → there must be some way of rescaling

  - What if we had matrices of size > 2

- The Wigner's surmise gives us the distribution of the spectral lines obtained

- Striking features of the above distribution.



even though x_1, x_2, x_3 are independent, the eigenvalues λ_1 & λ_2 talk to each other. λ_1 & λ_2 don't like to take same value. $P(s=0) = 0$.

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- distribution of spacing between cars and spacing between birds on a wire closely follow the Wigner surmise.

- Let us compare this distribution with the spacing between iid random variables.

$\{x_1, \ldots x_N\}$ → set of rvs.

joint distribution of all these variables is

$$P(x_1, \ldots x_N) = P_x(x_1)\, P_x(x_2) \ldots P_x(x_N)$$

- Let x have some distribution
- If we do a large number of random tosses of x, and calculate the spacing distribution after sorting x, what do we anticipate?

  - ~~we~~ ~~If~~ there are peaks in the $P_x(x)$, the tosses tend to cluster around this region and $P(s=0)$ is actually pretty high.

  - This is markedly different from our eigenvalue spacings, where the $P(s=0) \sim 0$

⇒ The eigenvalues are correlated and talk to each other, unlike the iid variables. even though the elements of the RM are not.

- Deriving $P_N(s)$

probability density of having spacing s in N tosses.

- $P_N(s \mid X_j = x)$ = conditional PDF given that $X_j = x$, all other darts have landed outside the interval.

$\leftarrow$ $X_j = x$ $X_{j+1} = x+s$ $\rightarrow$

$$= P_X(x+s)\left[\underbrace{1 + F(x)}_{\text{CDF}} - F(x+s)\right]^{N-2}$$

CDF → prob that a variable has landed to left of X

prob that variable has landed to left of $X+S$

- $F(x+s) - F(x)$ → Prob of landing inside the interval
- $1-$ that → Prob of landing outside the interval.

- $P_N(s)$ any $X = x) = \sum_{j=1}^{N} P_N(s \mid X_j = x)\, \underbrace{P_X(X_j = x)}_{\text{the PDF of } X}$

$$= N\, P_N(s \mid X_j = x)\, P_X(x)$$

- $P_N(s) = \int_{\text{support}} dx\; P_N(s \mid \text{any } X = x)$

$$= \int N\, P_N(s \mid X_j = x)\, P_X(x)\, dx$$

- Local change of variables

$$s = \frac{\hat{s}}{N P_X(x)}$$

$\hat{s}$ is an order 1 quantity

- why? – If we toss a lot of darts $(N \to \infty)$, the typical spacing between darts ↓
  – If we toss at a spot of high $P_X(x)$, typical spacing between darts ↓

- we want to have a way to wash out these two effects and then study

$$P_N\left(s = \frac{\hat{s}}{NP_X(x)} \,\middle|\, x_j = x\right) = \underbrace{P_X\left(x + \frac{\hat{s}}{NP_X(x)}\right)}_{P_X(x) + P_X'(x)\frac{\hat{s}}{NP_X(x)}} \left[1 + F(x) - \underbrace{F\left(x + \frac{\hat{s}}{NP_X(x)}\right)}_{\approx F(x) + \frac{\hat{s}}{NP_X(x)} F'(x)}\right]^{N-1}$$

$$(N \to \infty) \approx P_X(x)\left[\exp(-\hat{s})\right]$$

↳ now integrate this w.r.t x

$$\hat{P}_N(\hat{s}) = \underbrace{P_N\left(s = \frac{\hat{s}}{NP_X(x)}\right)}_{= NP_X(x)\exp(-\hat{s})} \underbrace{\frac{ds}{d\hat{s}}}_{= \frac{1}{NP_X(x)}}$$ (law of change of variables by probabilities)

$$= \frac{N \cdot 1}{N}\int \frac{P_X(x)}{P_X(x)} \exp(-\hat{s})\, dx \cdot NP_X(x)$$

$$= \boxed{\exp(-\hat{s})}$$