Basic Radar 3: Statistical Properties of Radar Signals

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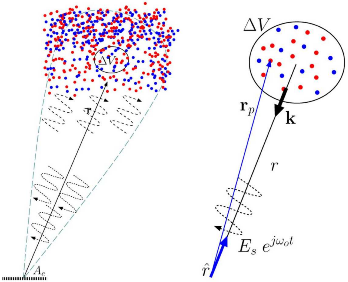
The Need for Statistical Descriptions of ISR Signals

If I knew the positions of every single electron in the scattering volume, I would know the received voltage exactly:



Exact expression for scattered electric field as a superposition of Thompson scatterers:

$$E_s = -\frac{r_e}{r} E_0 \sum_{p=1}^{N_0 \Delta V} e^{j\mathbf{k}\cdot\mathbf{r}_p}$$

ISR theory predicts statistical aspects of the scattered signal:

Scattered Power: $\left\langle |E_s|^2 \right\rangle$ Autocorrelation Function: $\langle E_s(t) E_s^*(t-\tau) \rangle$

These statistical properties are functions of macroscopic properties of the plasma: N_e, T_e, T_i, u_{los}.

Random Variables

A **random variable** is a variable whose numerical value depends on the outcome of a probabilistic phenomenon.

Probability Density Function:

$$P\left(x_1 < X < x_2\right) = \int_{x_1}^{x_2} p_X(x)\, dx$$

Expected Values:

$$E\left\{g\left(X\right)\right\} = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx$$

Mean:

$$Mean\left\{X\right\} = E\left\{X\right\}$$

Variance:

$$Var\left\{X\right\} = E\left\{\left(X - E\left\{X\right\}\right)^2\right\} = E\left\{X^2\right\} - \left(E\left\{X\right\}\right)^2$$

Collections of Random Variables

Multiple RVs must be described by joint-PDFs:

$$P\left(x_0 < X < x_1 \cup y_0 < Y < y_1\right) = \int_{x_0}^{x_1} \int_{y_0}^{y_1} p_{XY}\left(x, y\right)\, dydx$$

If X and Y are **independent**:

$$p_{XY}(x, y) = p_X(x) p_Y(y) \qquad p_{X|Y}(x|y) = p_X(x)$$

Relationships between RVs are defined through covariances:

$$Cov\left\{X, Y\right\} = E\left\{\left(X - E\{X\}\right)\left(Y - E\{Y\}\right)\right\}$$

Uncorrelated RVs have $Cov\{X, Y\} = 0$
Independent RVs are uncorrelated, but uncorrelated RVs are not necessarily independent.

Gaussian Distribution

A Gaussian random variable X has the following probability density function (Normal Distribution):

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x-\mu}{2\sigma^2}\right\}$$

$$E\{X\} = \mu \qquad Var\{X\} = \sigma^2$$

$$E\left\{(X-\mu)^4\right\} = 3\sigma^4$$

A jointly-Gaussian vector of random variables $\mathbf{X} = [X_0, X_1, X_2, \cdots, X_{N-1}]^T$ has the joint pdf:

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} |C|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}[\mathbf{x}-\mu]^T C^{-1} [\mathbf{x}-\mu]\right\}$$

$$E\{\mathbf{X}\} = \mu$$

$$Cov\{\mathbf{X}\} = E\left\{[\mathbf{X}-\mu][\mathbf{X}-\mu]^T\right\} = C$$

Properties of Jointly Gaussian Random Variables

- Linear combinations:

$$Z = \alpha X + \beta Y + \gamma \quad E\{Z\} = \alpha E\{X\} + \beta E\{Y\} + \gamma$$
$$Var\{Z\} = \alpha^2 Var\{X\} + \beta^2 Var\{Y\} + 2\alpha\beta Cov\{X, Y\}$$

- Matrix generalization:
  $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b} \quad E\{\mathbf{Y}\} = \mathbf{A}E\{\mathbf{X}\} + \mathbf{b} \quad Cov\{\mathbf{Y}\} = \mathbf{A}Cov\{\mathbf{X}\}\mathbf{A}^T$

Special cases for zero mean random variables:

- Odd moments are zero:
  $E\{V_1\} = E\{V_1 V_2 V_3\} = E\{V_1 V_2 V_3 V_4 V_5\} = \cdots = 0$
- Fourth moment theorem: $E\{V_1 V_2 V_3 V_4\} =$
  $E\{V_1 V_2\} E\{V_3 V_4\} + E\{V_1 V_3\} E\{V_2 V_4\} + E\{V_1 V_4\} E\{V_2 V_3\}$
- General even moment theorem (Isserlis' Theorem)
  $E\{V_1 V_2 \cdots V_{2n-1} V_{2n}\} = \sum \prod E\{V_i V_j\}$

Central Limit Theorem

Given a set of finite-variance, independent and identically distributed RV, $[X_0, X_1, \cdots, X_{K-1}]$, the distribution function of the average:

$$\hat{X} = \frac{1}{K} \sum_{n=0}^{K-1} X_n$$

will asymptotically approach a Gaussian distribution as K increases.

$$E\left\{\hat{X}\right\} = E\left\{X_n\right\} \qquad Var\left\{\hat{X}\right\} = \frac{1}{K} Var\left\{X_n\right\}$$

This is an amazingly useful theorem:

- Only the mean and variances of the intermediate quantities need to be calculated to predict the distribution of the final averaged result.
- Distribution functions of intermediate quantities do not need to be calculated in detail since the final averaged result will just be Gaussian.

Statistical Properties of ISR Voltages

Radar signals are complex-valued, zero-mean, Gaussian random vaiables with variances related to their power P:

$$\begin{aligned}
V &= V_R + jV_I \\
E\left\{V_R\right\} &= E\left\{V_I\right\} = 0 \\
E\left\{V_R^2\right\} &= E\left\{V_I^2\right\} = \frac{1}{2}P \qquad E\left\{V_R V_I\right\} = 0 \\
E\left\{|V|^2\right\} &= E\left\{V_R^2 + V_I^2\right\} = P \\
E\left\{V_R^4\right\} &= E\left\{V_I^4\right\} = \frac{3}{4}P^2 \qquad E\left\{V_R^2 V_I^2\right\} = E\left\{V_R^2\right\} E\left\{V_I^2\right\} = \frac{1}{4}P^2 \\
Var\left\{|V|^2\right\} &= E\left\{\left(|V|^2\right)^2\right\} - \left(E\left\{|V|^2\right\}\right)^2 \\
&= E\left\{V_R^4 + V_I^4 + 2V_R^2 V_I^2\right\} - \left(E\left\{V_R^2 + V_I^2\right\}\right)^2 \\
&= 2P^2 - P^2 = P^2
\end{aligned}$$

Power Estimation

Given K voltage samples with unknown signal power S, a known noise power N, and total power $P = S + N$, an estimate of the signal power is:

$$\hat{S} = \frac{1}{K}\sum_{n=0}^{K-1} |V_n|^2 - N$$

Expected Value: $E\left\{\hat{S}\right\} = \frac{1}{K}\sum_{n=0}^{K-1} E\left\{|V_n|^2\right\} - N = P - N = S$

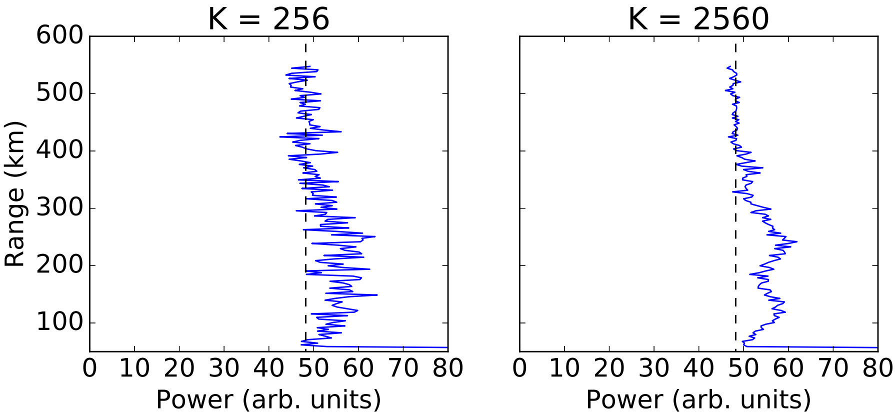
Variance (Invoke the Central Limit Theorem):

$$Var\left\{\hat{S}\right\} = Var\left\{\frac{1}{K}\sum_{n=0}^{K-1} |V_n|^2\right\} = \frac{1}{K} Var\left\{|V_n|^2\right\} = \frac{1}{K}P^2 = \frac{1}{K}(S+N)^2$$

Relative Error:

$$\frac{\sqrt{Var\left\{\hat{S}\right\}}}{S} = \frac{1}{\sqrt{K}}\frac{S+N}{S} = \frac{1}{\sqrt{K}}\left(1 + \frac{1}{S/N}\right)$$

Statistical Uncertainty and SNR are Different Concepts

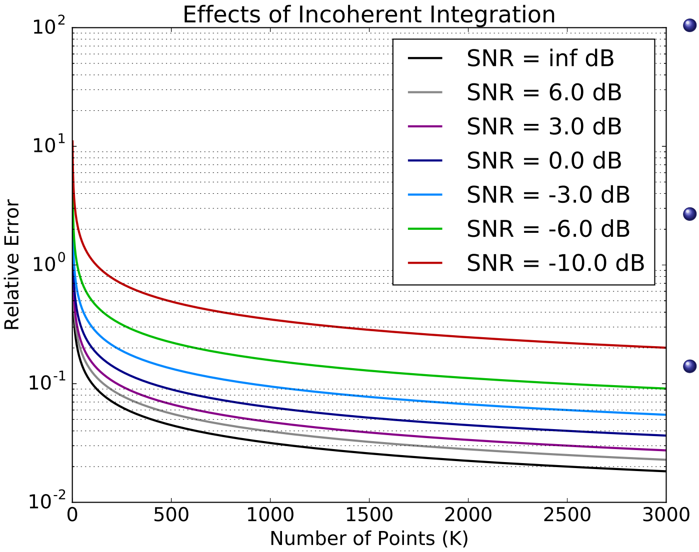


For $SNR = 0.25$:

$$K = 256 \rightarrow \text{Relative Error} = 31.25\%$$

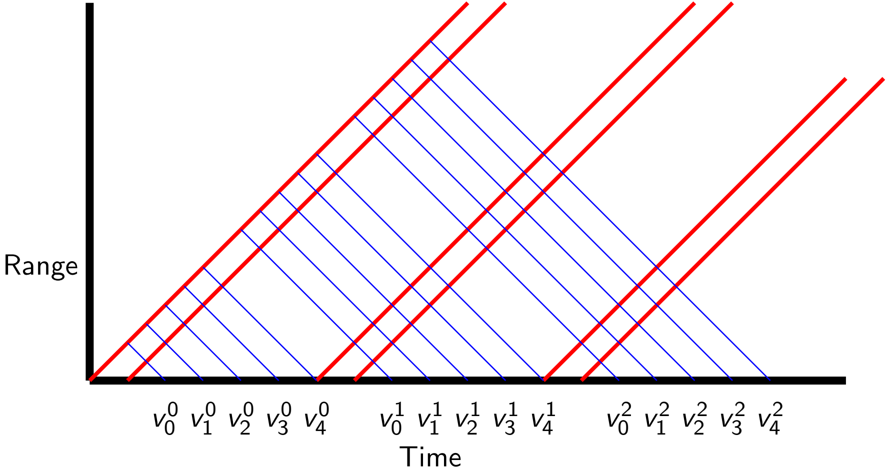
$$K = 2560 \rightarrow \text{Relative Error} = 9.88\%$$

Required Integration Times



- At $SNR = -3\ \mathrm{dB}$, 20% error requires $K = 225$.
- If the inter-pulse period is 5 ms, 225 pulses takes 1.125 s.
- If you cycle between 25 beams, 225 pulses in all beams takes 28.125 s.

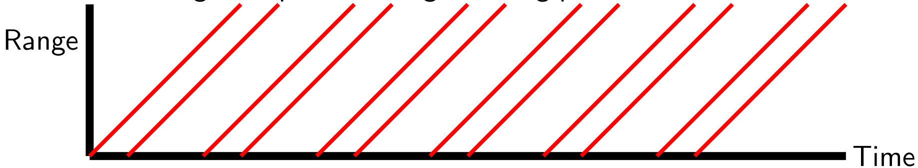
Problem with Short IPP: Range Aliasing



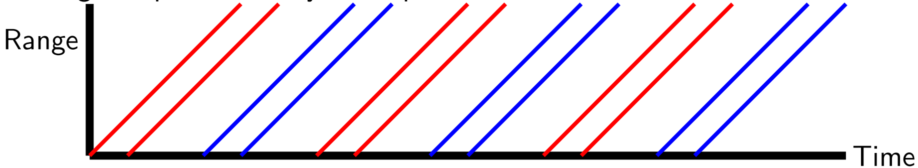
$$r_n = \frac{ct_n}{2} + m\frac{c\tau_{\mathrm{IPP}}}{2} \qquad \text{for any integer } m$$

Exploiting Frequency Diversity

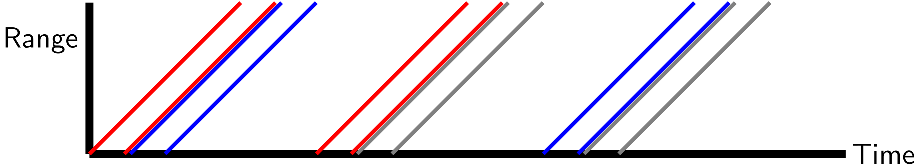
Pulses close together produce range aliasing problems:



Change frequencies every other pulse:



The RISR 3-frequency ImagingLP experiments:



Stochastic Processes: Definitions and Terminology

- Stochastic Process (aka Random Process): $V(t)$ where value at every time is a random variable
- Gaussian Stochastic Process:
  - PDF of each $V(t)$ is a Gaussian distribution (aka normal distribution)
  - Joint PDF of any subset of samples of $V(t)$ is a jointly Gaussian distribution (aka Multivariate Normal Distribution)
- Moments of a Stochastic Process:
  - Mean: $\bar{V}(t) = E\left\{V(t)\right\}$
  - Autocorrelation: $R_V\left(t, t-\tau\right) = E\left\{V(t)V^*(t-\tau)\right\}$
  - Autocovariance:
    $C_V\left(t, t-\tau\right) = E\left\{\left[V(t) - \bar{V}(t)\right]\left[V^*(t-\tau) - \bar{V}^*(t-\tau)\right]\right\} = R\left(t, t-\tau\right) - \bar{V}(t)\bar{V}^*(t-\tau)$
- (Wide Sense) Stationary Stochastic Process
  - $\bar{V}(t) = \bar{V}$ is independent of t
  - $R(t, t-\tau) = R(\tau)$ is independent of t
- ISR signals are Gaussian, zero mean, and stationary as long as the ionospheric state parameters are constant.

Power Spectra of Deterministic Signals

Given a signal $f(t)$ and its fourier transform $F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$, the power spectrum is:

$$
\begin{aligned}
S_F(\omega) = |F(\omega)|^2 &= F^*(\omega)F(\omega) \\
&= \mathcal{F}\left\{f(-t') * f(t')\right\} \\
&= \mathcal{F}\left\{\int_{-\infty}^{\infty} f(t')f(t'-t)\,dt'\right\}
\end{aligned}
$$

When you filter a signal:

$$
\begin{aligned}
g(t) &= h(t) * f(t) \\
G(\omega) &= H(\omega)F(\omega) \\
S_G(\omega) &= |H(\omega)|^2\, S_F(\omega)
\end{aligned}
$$

Power Spectra of Stochastic Signals

Fourier transforms of stationary random processes do not exist.
Fourier transforms of ACFs will exist, and are the power spectra:

$$S_V(\omega) = \int_{-\infty}^{\infty} R_V(\tau) e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} E\{V(t)V^*(t-\tau)\} e^{-j\omega\tau}\, d\tau$$

Properties:

- $S(\omega)$ is real and $S(\omega) \geq 0$
- Short correlation times $\leftrightarrow$ wide bandwidth and vice versa
- $\int_{-\infty}^{\infty} S_V(\omega)\, d\omega = R(0) = E\{|V|^2\}$ (total power)
- If U = h * V, $S_U(\omega) = |H(\omega)|^2 S_V(\omega)$

Intuitive interpretation: $\int_{\omega_1}^{\omega_2} S_V(\omega)\, d\omega$ is the power in the frequency band from ω_1 to ω_2.

Example: Running Average of White Noise

Continuous white noise:

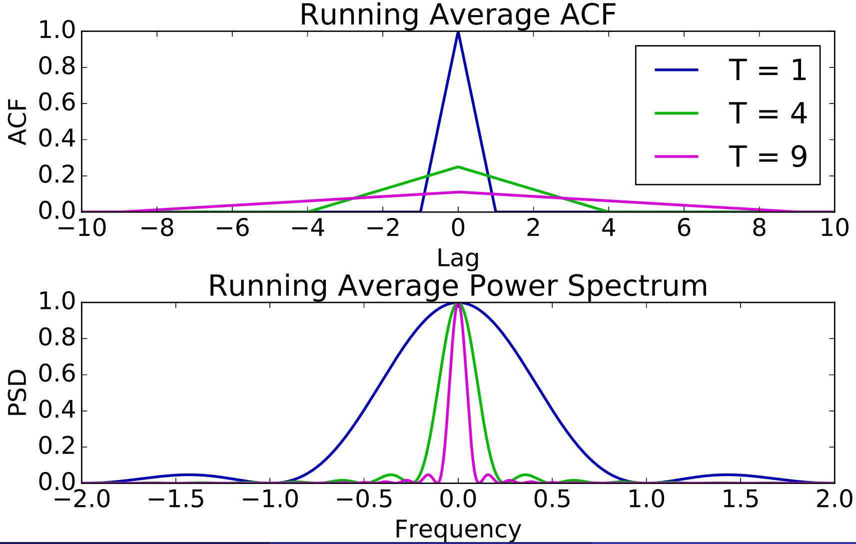
$$E\left\{W(t)\right\} = 0 \qquad S_W(\omega) = S_0 \qquad R_W(\tau) = S_0 \delta(\tau)$$

Running average of white noise:

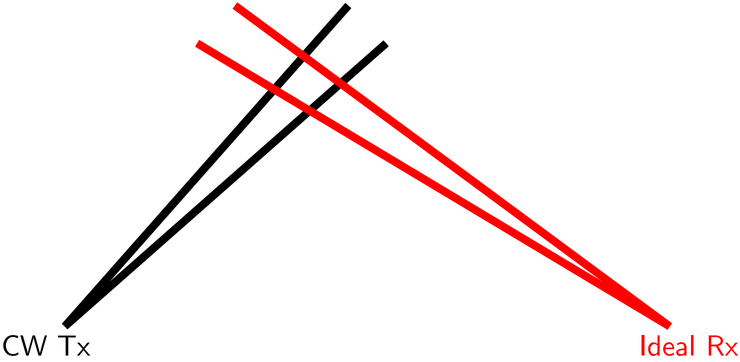
$$\begin{aligned}
V(t) &= \frac{1}{T} \int_{t-T/2}^{t+T/2} W(t')\, dt' \\
R_V(\tau) &= E\left\{ \frac{1}{T} \int_{t-T/2}^{t+T/2} W(t')\, dt' \frac{1}{T} \int_{t+\tau-T/2}^{t+\tau+T/2} W(t'')\, dt'' \right\} \\
&= \frac{1}{T^2} \int_{t-T/2}^{t+T/2} \int_{t+\tau-T/2}^{t+\tau+T/2} S_0 \delta\left(t' - t''\right)\, dt'' dt' \\
&= \begin{cases} S_0 \frac{T-|\tau|}{T^2} & |\tau| < T \\ 0 & |\tau| \geq T \end{cases} \quad \Rightarrow \quad S_V(\omega) = S_0 \left(\frac{\sin\left(\omega T/2\right)}{\omega T/2} \right)^2
\end{aligned}$$

Correlation Time and Bandwidth

Short correlation times → wide-bandwidth and vice versa



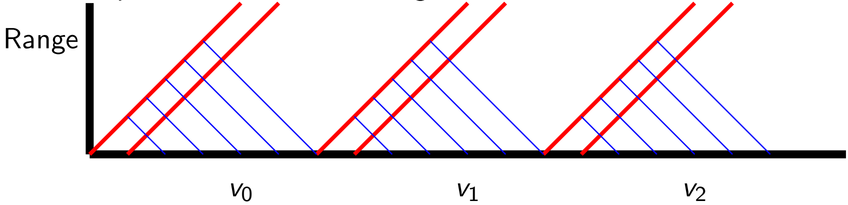
A Hypothetical CW Bistatic ISR Experiment



ISR theory gives the PSD and ACF of the received voltages as a function of N_e, T_e, T_i, and u_{los} in the overlap volume.

ACF Estimation (Pulse-to-Pulse)

Assume pulses are taken close together and are correlated.



Unbiased Estimator:

$$\hat{R}_\ell = \frac{1}{K-\ell}\sum_{n=\ell}^{K-1} v_n v_{n-\ell}^*$$

$$E\left\{\hat{R}_\ell\right\} = R_\ell$$

Biased Estimator:

$$\tilde{R}_\ell = \frac{1}{K}\sum_{n=\ell}^{K-1} v_n v_{n-\ell}^*$$

$$E\left\{\tilde{R}_\ell\right\} = \frac{K-\ell}{K} R_\ell$$

Zero-padded Periodogram

$$\tilde{S}_n = \left|\sum_{k=0}^{K-1} v_k e^{-2\pi j \frac{nk}{2K}}\right|^2$$

$$\tilde{R}_\ell = \frac{1}{2K}\sum_{n=0}^{2K-1} \tilde{S}_n e^{2\pi j \frac{n\ell}{2K}}$$

Biased ACF estimator equals the iFFT of the zero-padded periodogram.

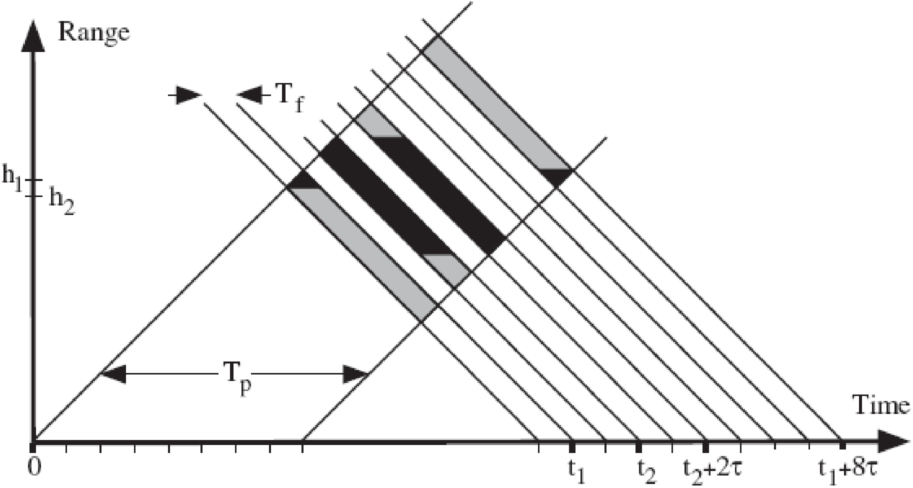
Underspread vs Overpread Targets

- If the IPP is short compared to the correlation time of the signal (inverse bandwidth), pulse-to-pulse processing works great.
- If the IPP is long compared to the correlation time, all pulse-to-pulse lag products give ≈ 0.
- Shortening the IPP is not always an option due to range aliasing.

Terminology:

- **Underspread target**: There exists an IPP that is short compared to the correlation time but long enough to avoid range aliasing.
  - D-region ISR
  - Perpendicular to **B** ISR
  - MST radar
- **Overspread target**: All practical IPP are long compared to the correlation time.
  - Most ISR experiments
  - SuperDARN

Uncoded Long Pulse Experiments



Scattered signals from outside the overlap region do not affect the expected value of a lag product, but they do affect the variance

Range-Lag Ambiguity Functions

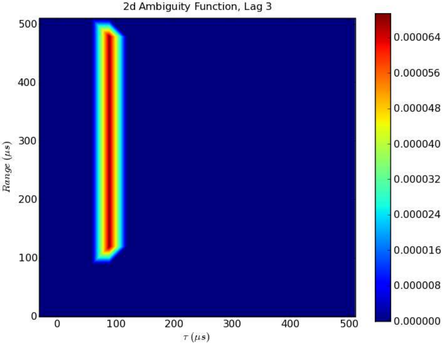
$$E\left\{v(t_2)v^*(t_1)\right\} = R_N(\tau) + \int d\tau dr\, R_S(r,\tau)\, W(r,\tau;t_1,t_2)$$

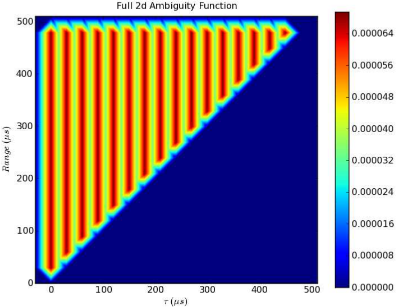
$$W(r,\tau;t_1,t_2) \equiv \int dt\, s\left(t+\tau-\frac{2r}{c}\right) h^*(t_2-t-\tau)\, s^*\left(t-\frac{2r}{c}\right) h(t_1-t)$$

- $R_N(\tau)$ is the noise ACF
- $R_S(r,\tau)$ is the theoretical signal ACF as a function of $N_e(r)$, $T_e(r)$, $T_i(r)$, $u_{\mathrm{los}}(r)$
- s is the envelope of the transmitted waveform
- h is the receiver impulse response
- $W(r,\tau;t_1,t_2)$ describes how different ranges and lags blur together when taking the variance of two samples

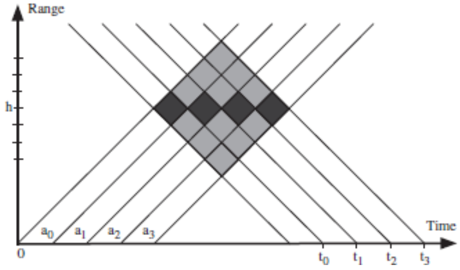
2-D Range-lag Ambiguity Function of Long Pulse

Ambiguity function with a boxcar filter. 480 μs long pulse, 30 μs sampling.





Coded Pulse Experiments

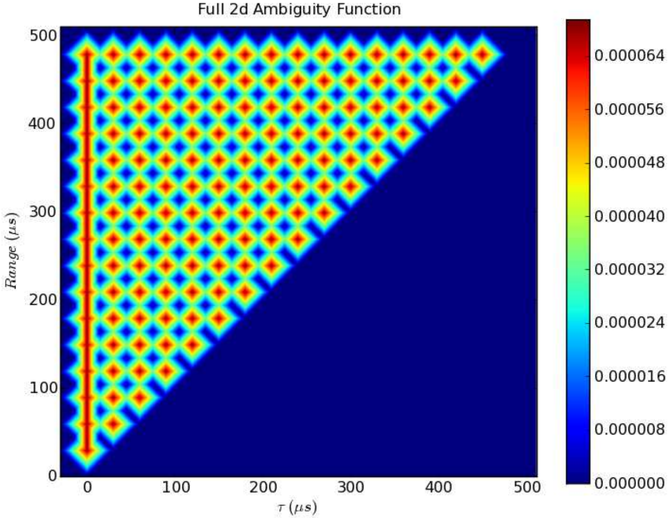


$$a_0 a_1 v_0 v_1^* = a_0\left(a_0 s_h^t + a_1 s_{h-1}^{t+\frac{1}{2}} + a_2 s_{h-2}^{t+1} + a_3 s_{h-3}^{t+\frac{3}{2}}\right)\times$$
$$a_1\left(a_0 s_{h+1}^{t+\frac{1}{2}} + a_1 s_h^{t+1} + a_2 s_{h-1}^{t+\frac{3}{2}} + a_3 s_{h-2}^{t+2}\right)^*$$

$$E\left\{a_0 a_1 v_0 v_1^*\right\} = E\left\{s_h^t s_h^{*t+1}\right\} + a_0 a_2 E\left\{s_{h-1}^{t+\frac{1}{2}} s_{h-1}^{*t+\frac{3}{2}}\right\}$$
$$+ a_0 a_1 a_2 a_3 E\left\{s_{h-2}^{t+1} s_{h-2}^{*t+2}\right\}$$

Range-lag Ambiguity Function of Alternating Codes

Ambiguity function for a boxcar filter. 480 μs (16-baud, 30 μs baud, 32 pulse).



ISR Analysis

1. After pulse k, take N samples $v_0^k, v_1^k, \ldots, v_{N-1}^k$
   - Each v_n^k will be complex-valued, Gaussian, and zero mean
2. Tabulate the products between samples $L_{n,m}^k = v_n^{k*} v_m^k$
   - Ambiguity function theory gives the expected value of each of these lag products as a function of the plasma parameters and the known s and h.
   - 4^{th} moment theorem can be used to derive (Hysell et al. [2008]):

$$Var\left\{\Re\left(L_{n,m}^k\right)\right\} = \frac{1}{2}\Re\left(E\left\{L_{n,n}^k\right\}E\left\{L_{m,m}^k\right\} + E\left\{L_{n,m}^k\right\}E\left\{L_{n,m}^k\right\}^*\right)$$

$$Var\left\{\Im\left(L_{n,m}^k\right)\right\} = \frac{1}{2}\Re\left(E\left\{L_{n,n}^k\right\}E\left\{L_{m,m}^k\right\} - E\left\{L_{n,m}^k\right\}E\left\{L_{n,m}^k\right\}^*\right)$$

3. Average over many pulses $\hat{L}_{n,m} = \frac{1}{K}\sum_{k=0}^{K-1} L_{n,m}^k$
   - By the central limit theorem, real and imaginary $\hat{L}_{n,m}$ will be Gaussian and their variances will scale as $1/K$
4. Real and imaginary parts of $\hat{L}_{n,m}$ and their variances are inputs to fitting algorithm

Error Propagation Through the ISR Processing Chain