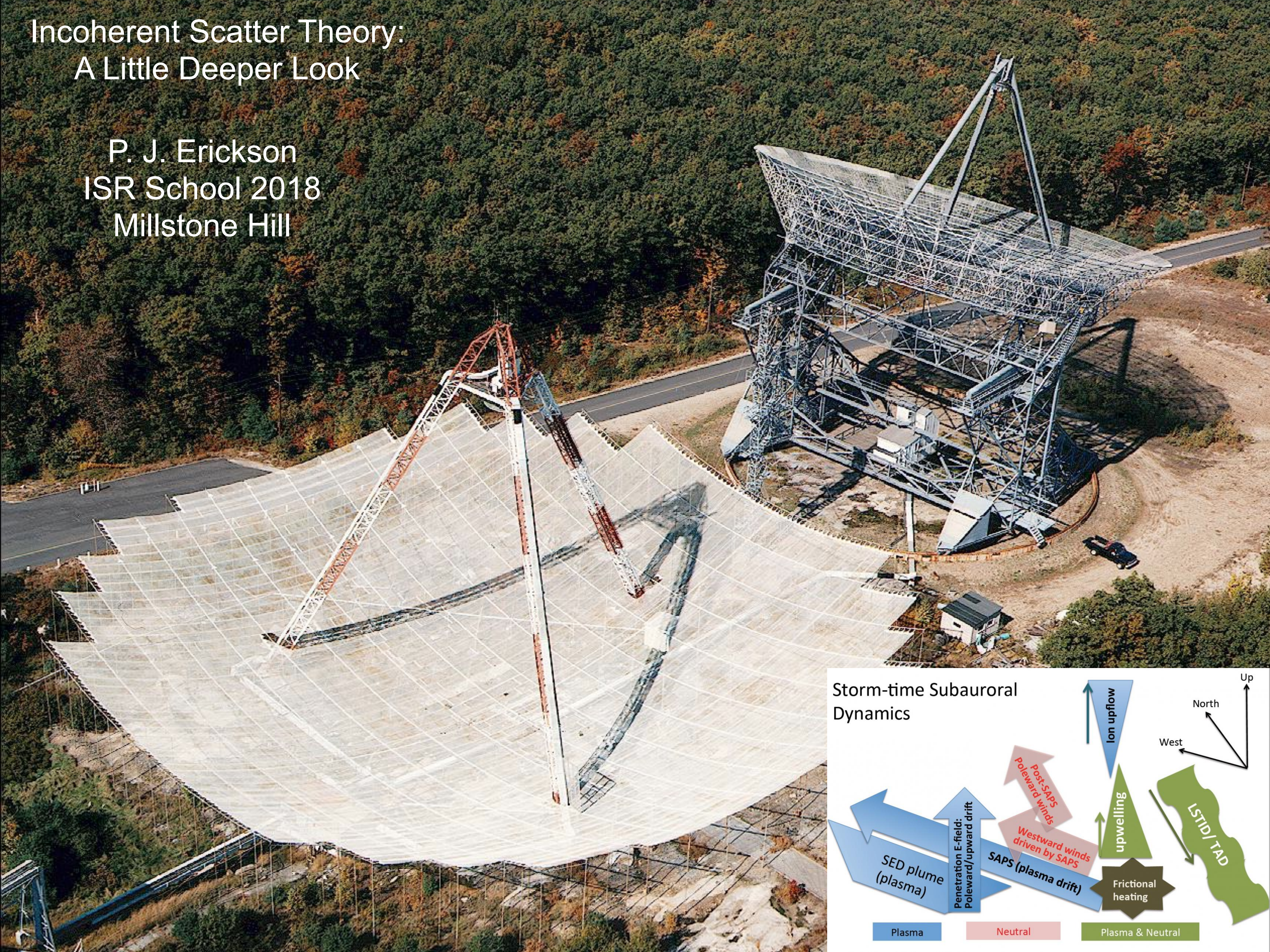
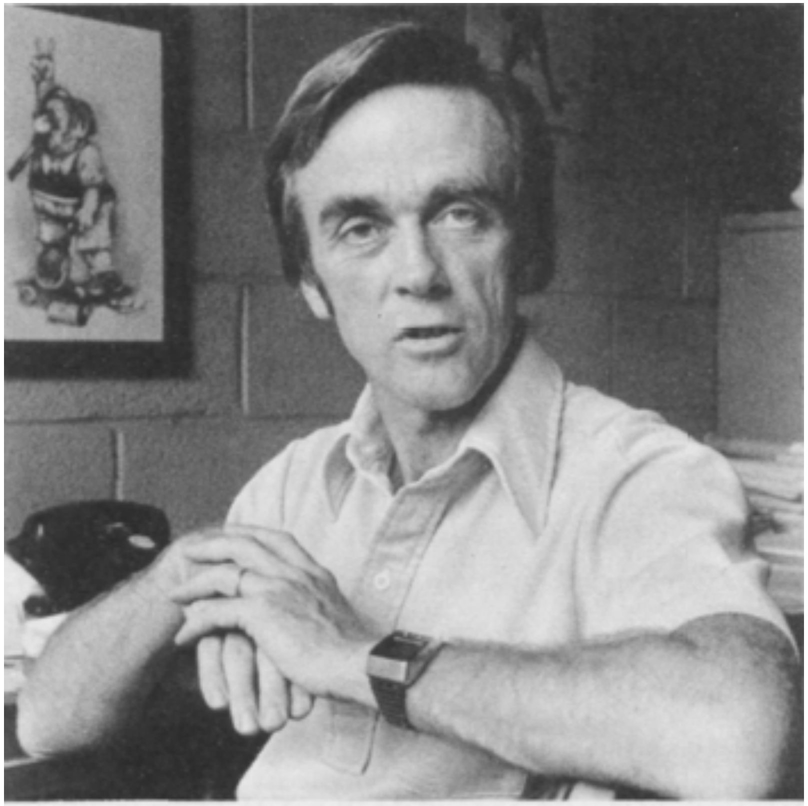
Incoherent Scatter Theory: A Little Deeper Look

P. J. Erickson
ISR School 2018
Millstone Hill





A theory of incoherent scattering of radio waves by a plasma

By J. P. Dougherty and D. T. Farley†

Cavendish Laboratory, University of Cambridge

(*Communicated by J. A. Ratcliffe, F.R.S.—Received* 10 *May* 1960—
Revised 23 *June* 1960)

A theory is developed which describes the scattering of radio waves by the random thermal fluctuations of electron density in a collision-free plasma. The frequency spectrum, as well as the amplitude, of the scattered radiation is calculated. Particular attention is paid to the part of the spectrum which corresponds to small Doppler shifts, this being the region of greatest significance in connexion with the phenomenon of incoherent scattering from the ionosphere.

The calculations are based on a generalized version of Nyquist's noise theorem, and they lead to the following conclusions:

(1) The mean scattering cross-section for the ionosphere is equal to that which would exist if each of the electrons scattered independently with a cross-section of one-half the classical Thomson cross-section.

(2) The mean Doppler broadening of the scattered signal corresponds roughly to the speed of the ions rather than to that of the electrons.

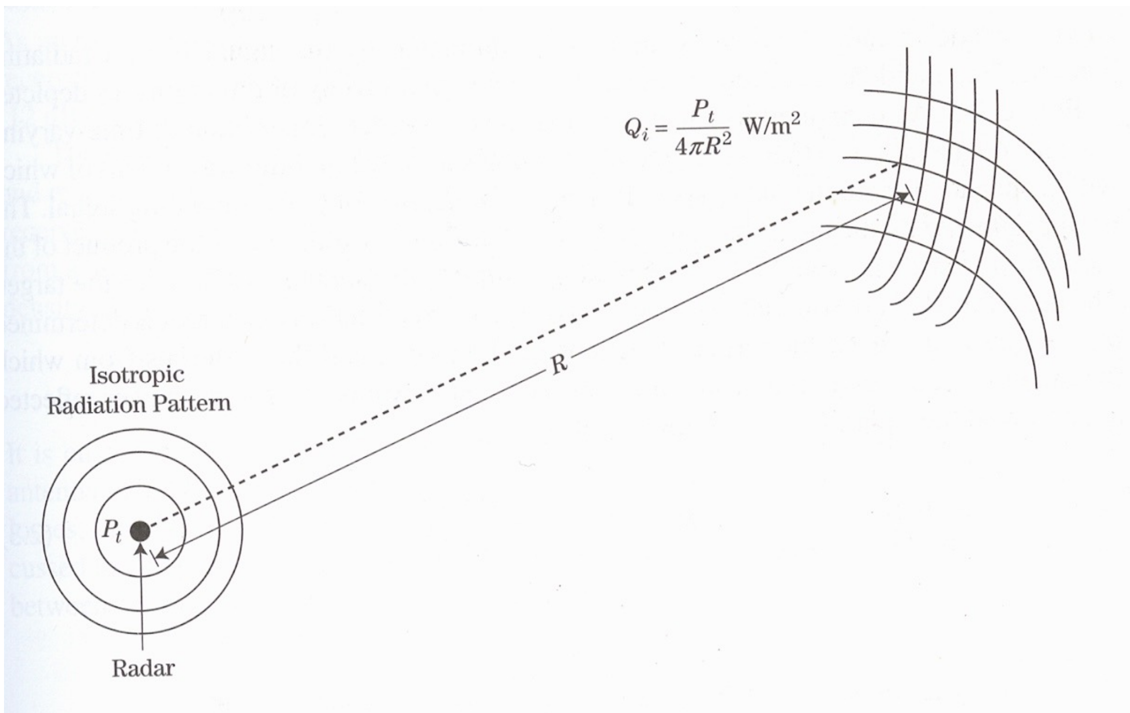
(3) The spectral shape of this signal is not Gaussian. There is a mild maximum in the spectrum away from the central frequency, as can be seen in figure 1.

(4) Plasma resonance effects contribute only negligibly to the scattering for frequencies currently of interest.

Donald T. Farley
1933-2018

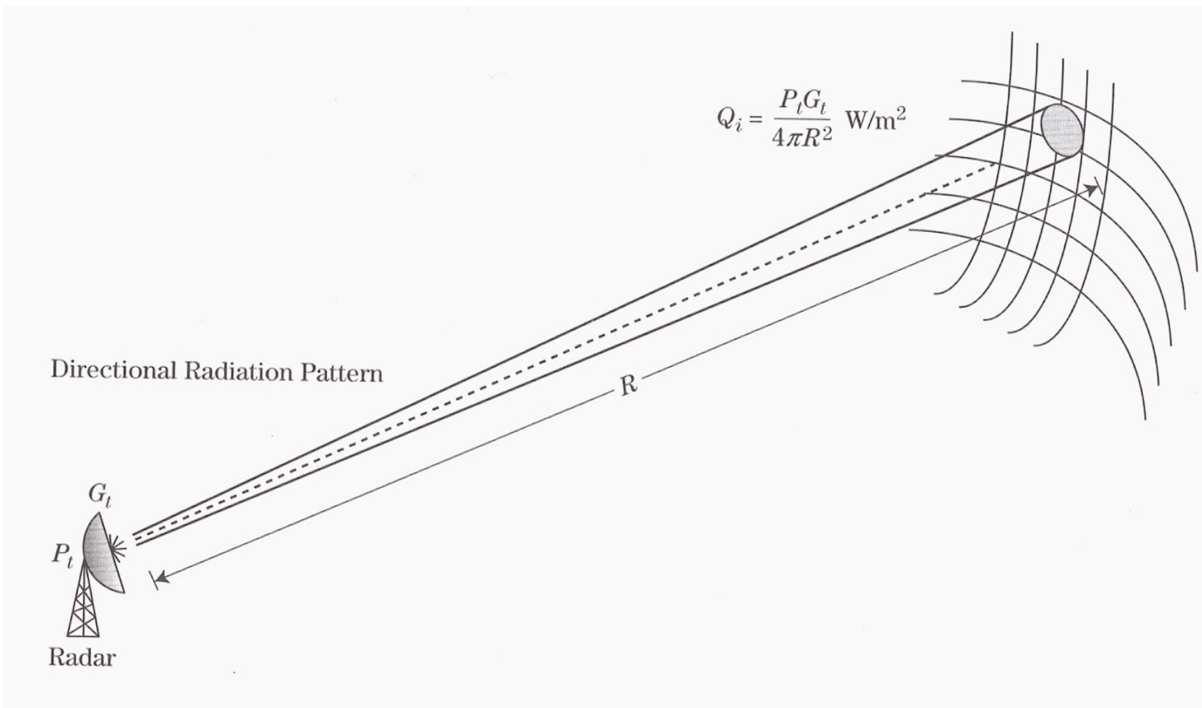
The Radar Equation: Monostatic Version

Power density at range R (isotropic): $$\frac{P_t}{4\pi R^2}$$



The Radar Equation: Monostatic Version

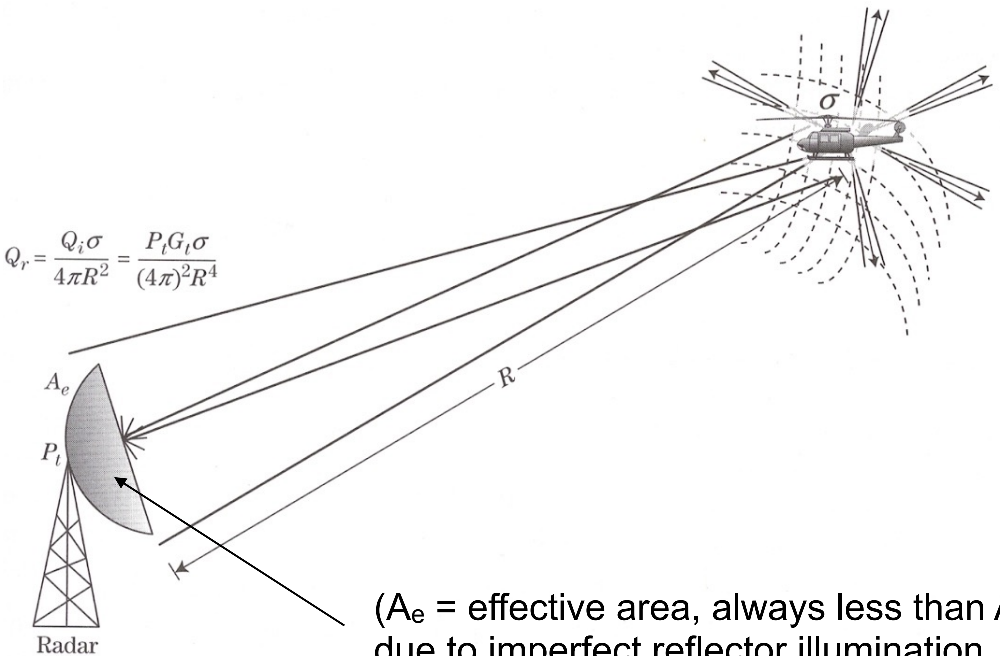
Power density at range R (directional): $$\frac{P_t\ G}{4\pi R^2}$$



The Radar Equation: Monostatic Version

Radar cross section (m^2)

Reradiated power density at Rx: $$\frac{P_t\ G}{4\pi R^2} \cdot \frac{\sigma}{4\pi R^2}$$



(A_e = effective area, always less than A due to imperfect reflector illumination, spillover, etc.)

The Radar Equation: Monostatic Version

Total received power: $$P_r = \frac{P_t\ G}{4\pi R^2} \cdot \frac{\sigma}{4\pi R^2} \cdot A_e = \frac{P_t\ G\ A_e\ \sigma}{(4\pi)^2 R^4}$$

Use gain/area relation - $$\boxed{A_{eff} = G\ A_{eff,iso} = G\frac{\lambda^2}{4\pi}}$$

The Radar Equation:

$$P_r = P_t \frac{A_e^2}{4\pi\lambda^2 R^4}\sigma$$

Hard vs Soft Radar Targets

Generalize radar equation for one or more scatterers, distributed over a volume:

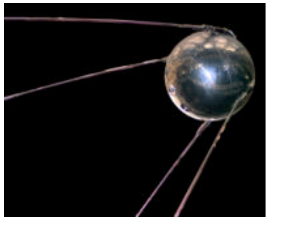
$$P_r = \int P_t \, \frac{A_e^2}{4\pi\lambda^2 R^4} \, \eta(\vec{x}) \, dV_s$$

First case: single scatterer ("hard target") at single point in space:

$$\int \eta(\vec{x}) \, dV_s = \sigma_{target} \equiv \sigma$$

Hard target radar equation:

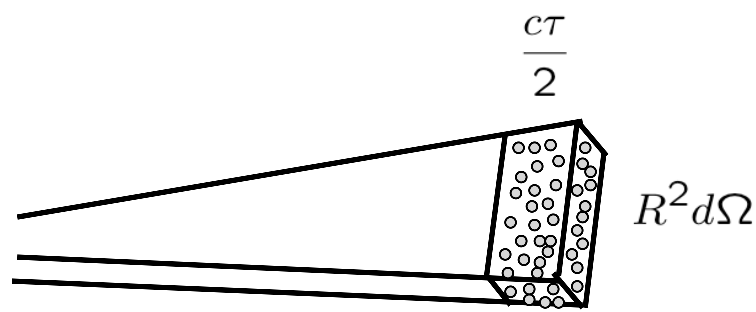
$$P_r = P_t \frac{A_e^2}{4\pi\lambda^2 R^4} \sigma$$



Sputnik 1 (1957-10-04)

Distributed Targets

$$\int \eta(\vec{x})\ dV_s = \int_0^{2\pi} \int_0^{\pi} \eta(\vec{x})\ \frac{c\tau}{2}\ R^2\ \sin\theta\ d\theta\ d\phi$$



Assume volume is filled with identical, isotropic scatters

scattering cross-section of one electron

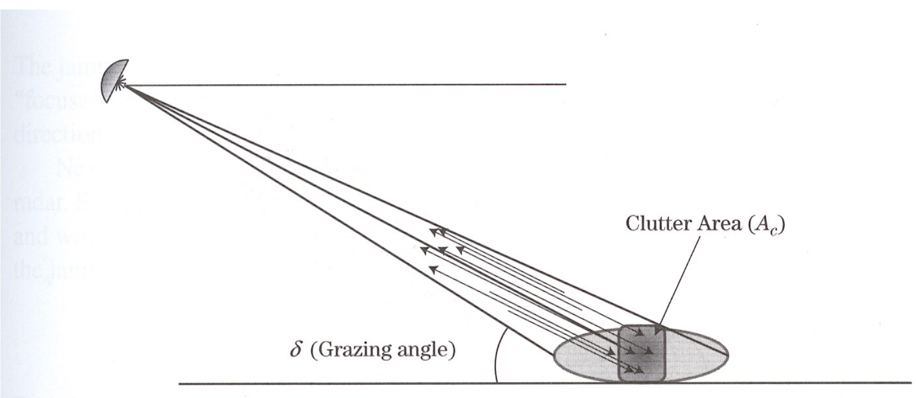
$$\int \eta(\vec{x})\ dV_s = \frac{c\tau}{2}\ R^2\ \sigma_e$$

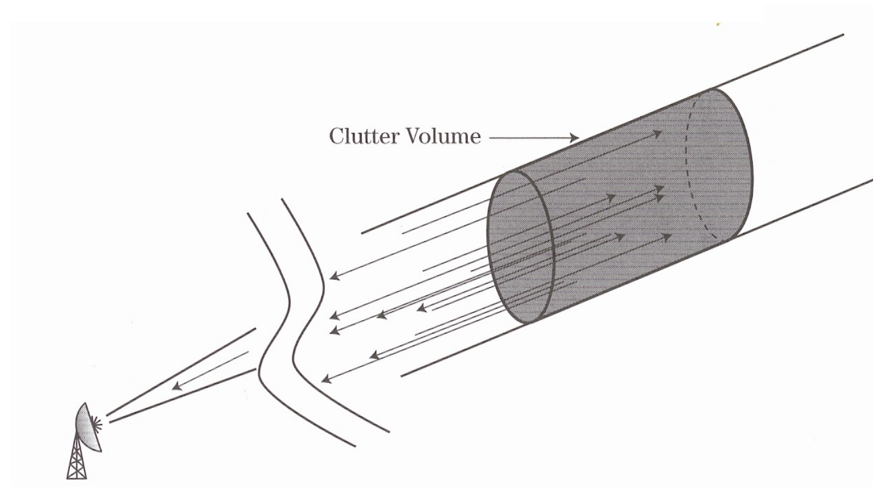
Distributed Scatterers

$$P_r = P_t \frac{A_e^2}{4\pi\lambda^2 R^4} \sigma_e \frac{c\tau}{2} R^2$$

The "soft target" Radar Equation

$$\int \eta(\vec{x})\, dV_s = \frac{c\tau}{2} R^2 \sigma_e$$

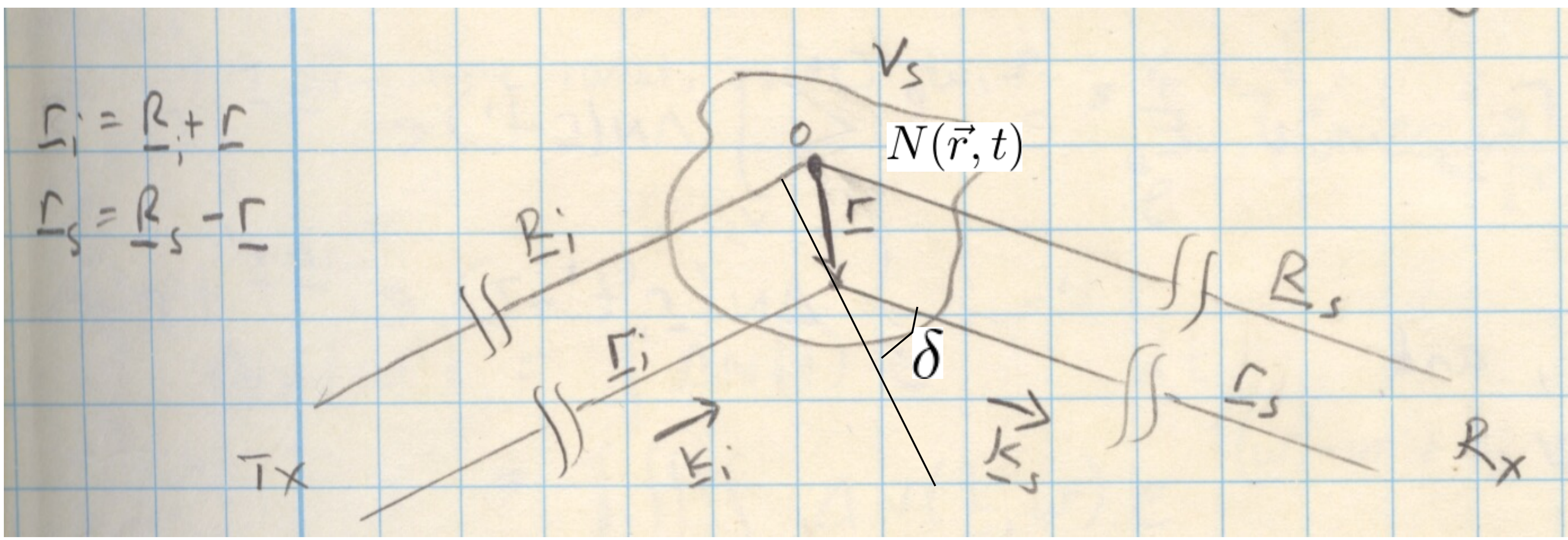




Single Electron Scattering Model

Incident EM wave accelerates each charged particle it encounters. These then re-radiate an EM wave (as Hertzian dipoles).

For a single electron located at r = 0, we need the scattered field at a distance r_s.



Single Electron Scattering Model

Incident EM wave accelerates each charged particle it encounters.
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For a single electron located at r = 0, the scattered field at a distance r_s :

scattered field

$$\left|\vec{E}_s(\vec{r}_s, t)\right| = \frac{e^2 \mu_0 \sin\delta}{4\pi r_s m_e} \left|\vec{E}_i(0, t^{'})\right|$$

Incident field

$$= \frac{r_e}{r_s} \sin\delta \left|\vec{E}_i(0, t')\right|$$

$$r_e = \frac{e^2 \mu_0}{4\pi m_e}$$

Classical electron radius

$$t' = t - \frac{r_s}{c}$$

Delayed time

$$\sin\delta$$

Scattering angle

Scattering Model

Assume a volume filled with electron scatterers whose density is represented in space and time by

$$N(\vec{r}, t)$$

Illuminating this volume with an incident field from a transmitter location means that each electron contributes to the resulting scattered field, using *Born approximation* (each scatter is weak and does not affect others).

With geometrical considerations, scattered field at receiver location is now:

$$E_s(t) = r_e \sin\delta \; E_0 e^{j\omega_0 t''} \int_{V_s} \frac{1}{r_s} N(\vec{r}, t') e^{-j(\vec{k_i}-\vec{k_s})\vec{r}} d^3\vec{r}$$

$$t'' = t' - \frac{r_i}{c}$$

Delayed time (TX to RX)

$$t' = t - \frac{r_s}{c}$$

Delayed time (volume to RX)

Scattering Model

Assume densities have random spatial and temporal fluctuations about a background:

$$N(\vec{r},t) \rightarrow N_0 + \Delta N(\vec{r},t)$$

Further, assume backscatter (i.e. monostatic radar):

$$\vec{k} = 2\vec{k}_i \qquad r_i \equiv r_s = R \qquad \sin\delta \rightarrow 1$$

Then, scattered field reduces to:

$$E_s(t) \rightarrow \frac{r_e}{R}\, E_0 e^{j\omega_0 t''} \underbrace{\int_{V_s} \Delta N(\vec{r},t') e^{-j\vec{k}\cdot\vec{r}} d^3\vec{r}}_{\equiv \Delta N(\vec{k},t')}$$

Maxwell's Equations

J. C. Maxwell
1831 - 1879

Governs propagation of electromagnetic waves ("action at a distance"), relation between electric and magnetic field and motions of charges

Foundation of classical electromagnetic theory

in free space:
H = B D = E

Gauss' Law (electric field around charges)

$$\nabla \cdot \mathbf{D} = \rho_f$$

Gauss' Law for magnetism (no magnetic monopoles)

$$\nabla \cdot \mathbf{B} = 0$$

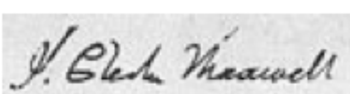
Faraday's Law (electric field around a changing magnetic field)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Ampere's Law (magnetic field circulation around electric charges)

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Maxwell's correction
(displacement current)



Scattering Model

Plasmas (ionosphere) are thermal gases and $\Delta N(\vec{r},t)$ is a Gaussian random variable, so the Central Limit Theorem applies:

statistical average

$$\langle E_s(t) \rangle = \langle \Delta N(\vec{r},t) \rangle = 0$$

It's much more useful to look at second order products – in other words, examine <u>temporal correlations</u> in the scattered field:

$$\langle E_s(t)\ E_s^*(t+\tau) \rangle \propto\ e^{-j\omega_0 \tau} \left\langle \Delta N(\vec{k},t)\ \Delta N^*(\vec{k},t+\tau) \right\rangle$$

Useful things to measure can now be defined.

Scattering: Measurable Quantities

Defining $C_s = \frac{r_e^2 E_0^2 \sin^2\delta}{R^2} V_s$, then

Total scattered power

$$\left\langle |E_s(t)|^2 \right\rangle = C_s \left\langle \left| \Delta N(\vec{k}) \right|^2 \right\rangle$$

and Autocorrelation function (ACF):

$$\langle E_s(t) E_s^*(t+\tau) \rangle = C_s e^{-j\omega_0 \tau} \left\langle \Delta N(\vec{k}, t) \Delta N^*(\vec{k}, t+\tau) \right\rangle$$

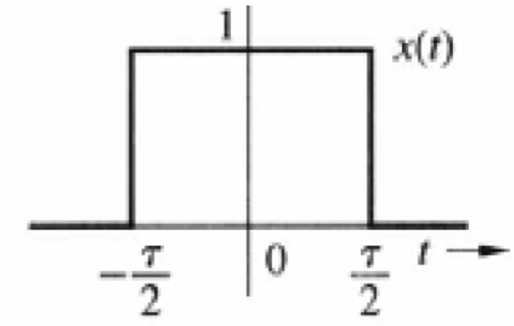
or Power Spectrum:

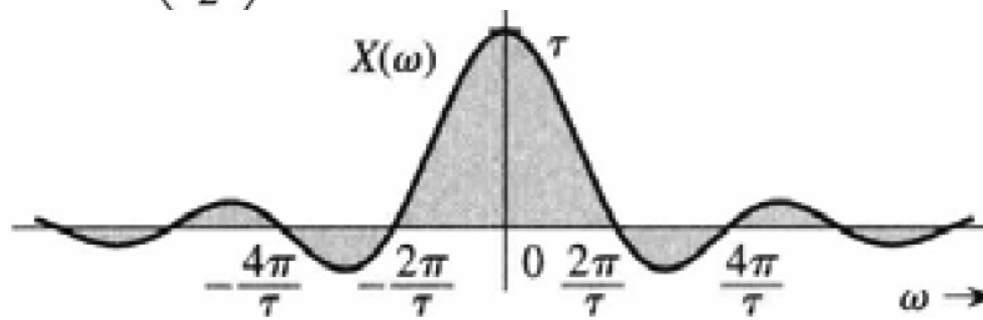
$$\left\langle |E_s(\omega_0 + \omega)|^2 \right\rangle \ \propto C_s \left\langle \left| \Delta N(\vec{k}, w) \right|^2 \right\rangle$$

The gate function and its Fourier transform

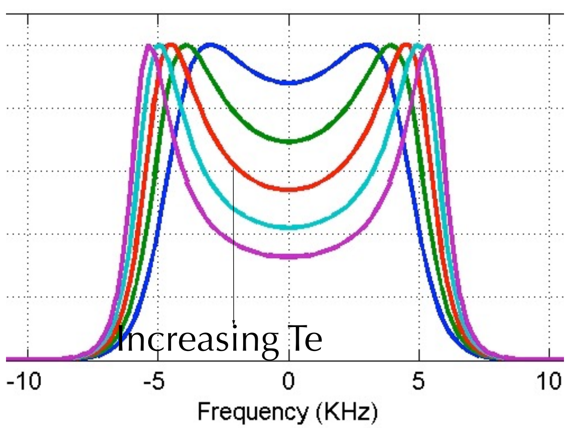
$$\mathrm{rect}(t/\tau) \;=\; \begin{cases} 1 & \text{for } -\tau/2 < t < \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

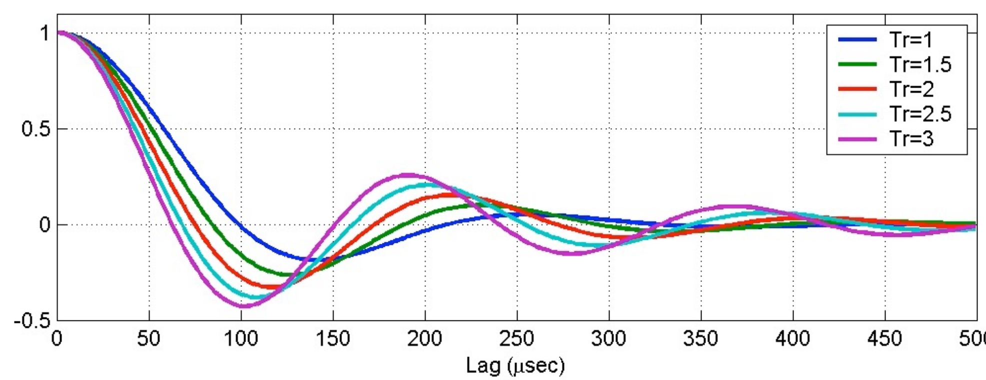
$$\mathrm{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$





ISR spectrum $\Longleftrightarrow$ Autocorrelation function (ACF)





Not surprisingly, the ISR ACF looks like a sinc function…

Incoherent Scattering Model: Summary

Radar filters in k space:

$$\Delta N(\vec{r},t) \rightarrow \Delta N(\vec{k}_r,t)$$

$$\Delta N(\vec{k}_r,t) \propto E_s(t)$$

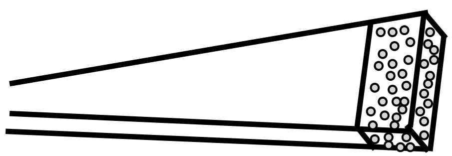
Form ACF of $E_s(t)$ for each range, average, transform to a power spectrum:

$$\langle E_s(t)E_s^*(t+\tau)\rangle \rightarrow \left\langle \left|\Delta N(\vec{k},w)\right|^2 \right\rangle$$

Interpret latter in terms of the medium parameters.

Remote sensing a plasma: The experimental (radar) view

Suppose we transmit a wave towards a plasma and measure the scattered wave:



$$P_{rec} = (P_{inc}) A_{scat} \left(\frac{A_{rec}}{4\pi R^2}\right)$$

$$A_{scat} = \sigma_{radar} V_s$$

(ionosphere is a beam filling target)

$$\sigma_{radar} = 4\pi \sigma_{total}$$

(Solid angle)

$$\left(\frac{P_{rec}}{P_{inc}}\right)\left(\frac{4\pi R^2}{A_{rec}}\right)\left(\frac{1}{V_s}\right) = 4\pi r_e^2 \sin^2\delta \left\langle |\Delta N(k)|^2 \right\rangle$$

Measurable experimentally

Physics info is here!

(1 for backscatter)

Radar cross-section of ionospheric plasma

Assume a beam filling plasma at F region altitudes (300 km) with very high electron density (1E12 electrons per m3 - BEST CASE):

Classical electron scattering cross-section $\sigma_e = 10^{-28} m^2 / e^-$

Assume a pulse length of 10 km.
Assume a cross-beam width of 1 km (~ Arecibo).

Total cross section is then (10 km x 1 km x 1 km x 1E12 m^-3 x 1E-28 m^-2/e-):

$$\boxed{\sigma_{tot} \sim 10^{-6} m^2}$$

-60 dBsm! Are we going to be able to do this at all?

NB: Born approximation is very valid, since total amount of scattered power in the volume ~ 1E-12. So we can make full range profiles if we can detect the scatter.

Detectability of scatter from ionospheric plasma

For fraction of scattered power actually received, assume isotropic scatter and a BIG ~100 m diameter antenna:

$$f_{rec} = \frac{A_{rec}}{4\pi R^2} \sim \frac{10^4 \ m^2}{4(300 \times 10^3 \ m)^2}$$

About -80 dB (1E-8): not much. So:

$$\frac{P_{rec}}{P_{tx}} \sim 10^{-20}$$

So a radar with 1 MW transmitted signal receives 10 femtowatts of incoherently scattered power from free electrons in the ionosphere.

REALLY not very much.

Detectability of scatter from ionospheric plasma

What matters, though, is the signal to noise ratio:

$$P_{noise} = (k_B T_{eff})\,(BW)$$

(derived later)

Typical effective noise temperatures ~100 to 200 K at UHF frequencies (430 MHz, say).

Assume the bandwidth is set by thermal electron motions in a Boltzmann sense:

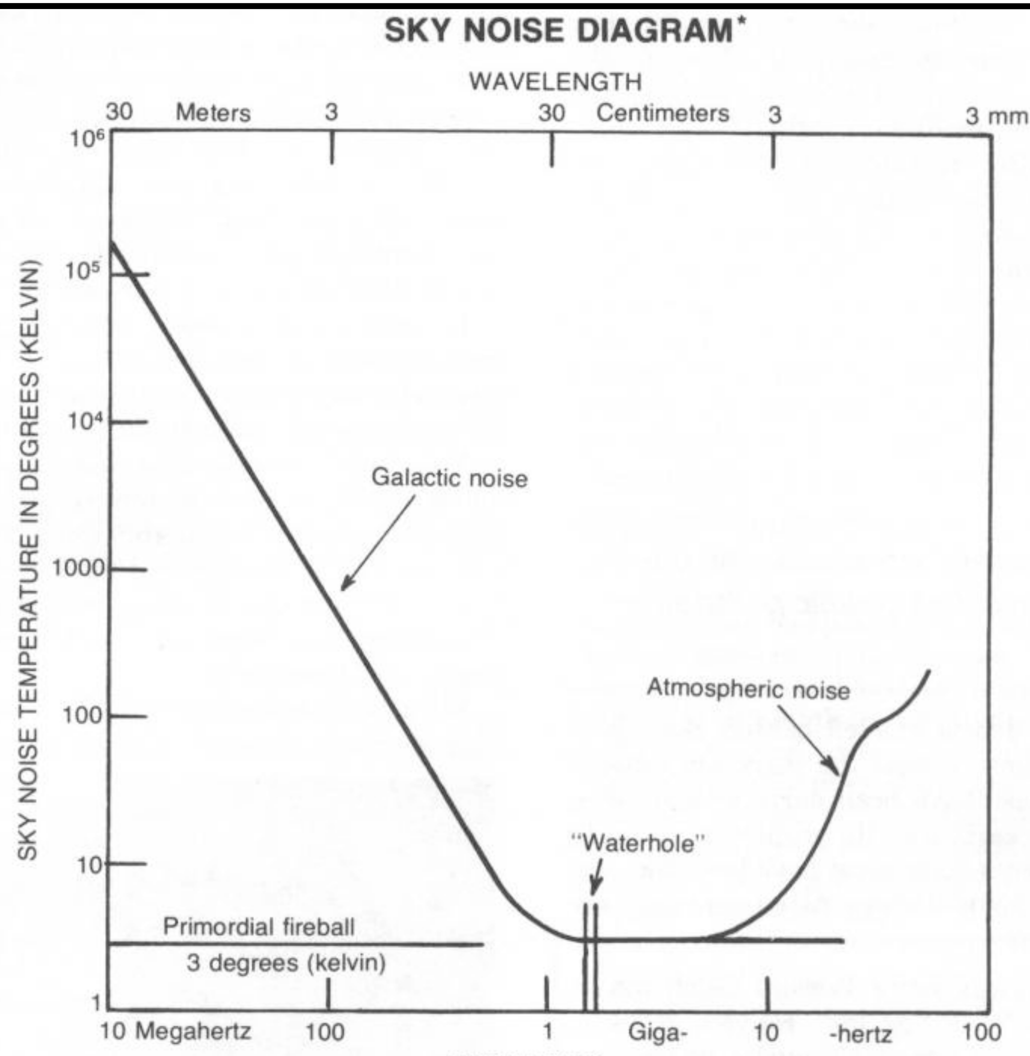
$$3k_B T_e \sim m_e v_{e,th}^2$$

$$v_{e,th} \sim \sqrt{\frac{3k_B T_e}{m_e}} \sim 2 \times 10^5 m/s$$

$$BW \sim (v_{e,th})\,(2)(2)(\frac{f_{tx}}{c}) \sim 10^6 Hz$$

(2s are for up/down, backscatter)

Sky Noise: The Universe Is Also Transmitting



*After J. D. Kraus, "Radio Astronomy", McGraw-Hill, 1966.

Nyquist-Johnson thermal noise (1928): Always present

JULY, 1928 PHYSICAL REVIEW VOLUME 32

THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

BY H. NYQUIST

ABSTRACT

The electromotive force due to thermal agitation in conductors is calculated by means of principles in thermodynamics and statistical mechanics. The results obtained agree with results obtained experimentally.

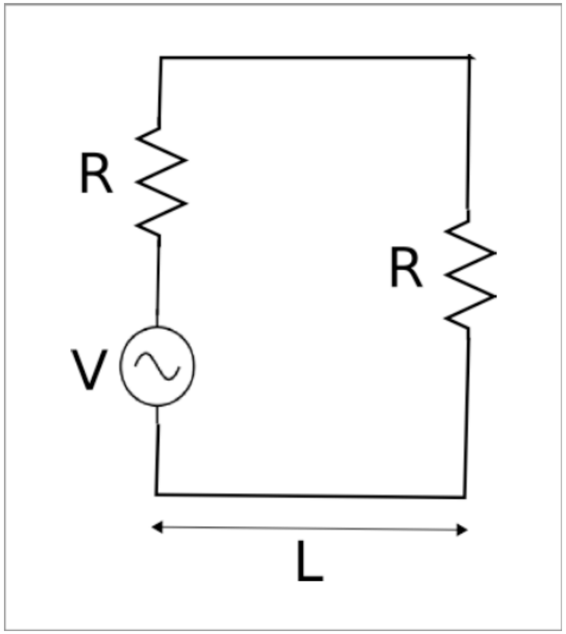
H. Nyquist 1889-1976
(born Nilsby, Sweden)

"Bert" Johnson 1887-1970
(born Gothenburg, Sweden)

Nyquist-Johnson: The Motivation

Nyquist-Johnson thermal noise (1928)

Circuit loop has a temperature T
(with thermal vibrations)



(Each resistor)

$$P_{absorbed} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \; \frac{h|\omega|}{e^{\frac{h|\omega|}{k_B T}} - 1}$$

$$I = \frac{V}{2R}$$

$$P_{emitted} = \frac{\langle V^2 \rangle}{4R}$$

$S(\omega)$

(Each resistor)

Voltage spectral density in the circuit loop (= per frequency):

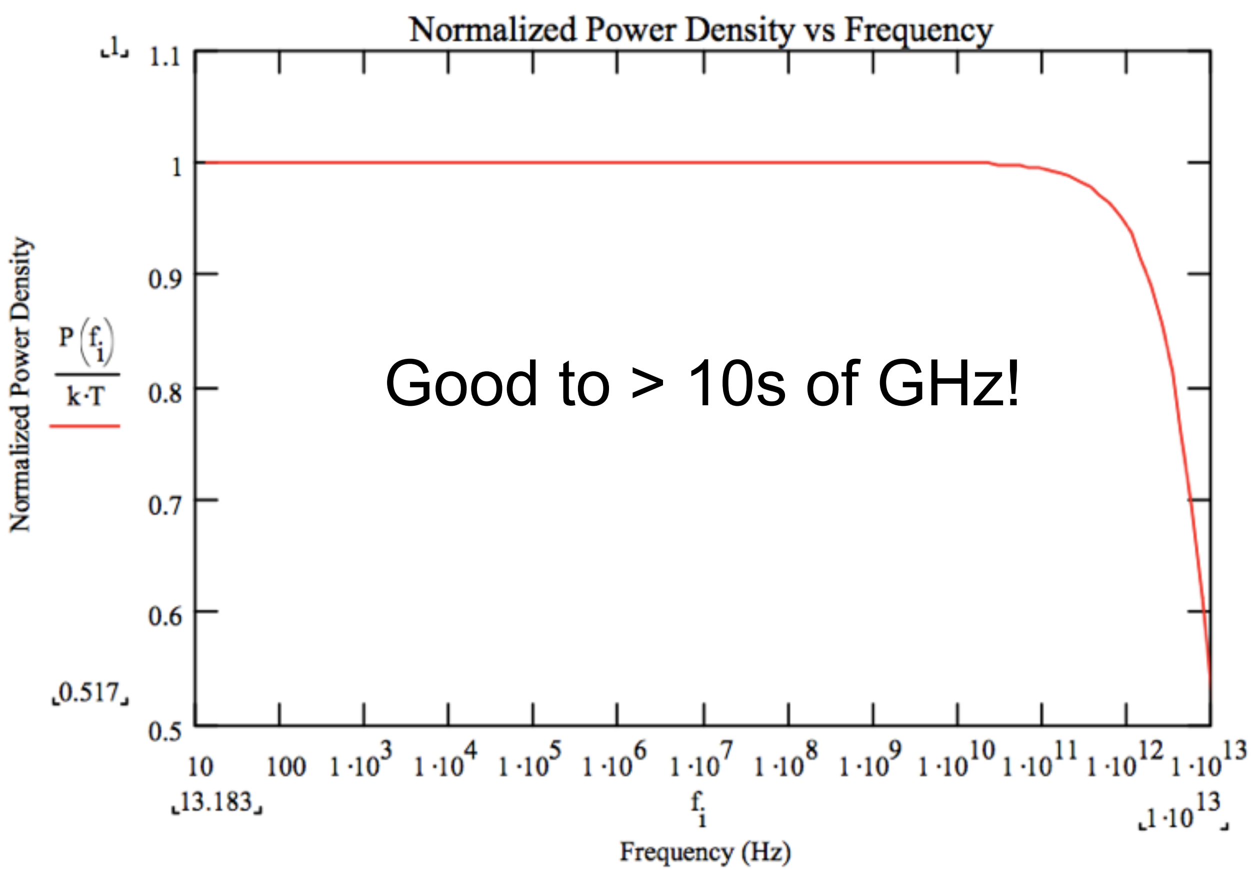
$$S(\omega) = 4R\frac{h|\omega|}{e^{\frac{h|\omega|}{k_B T}} - 1}$$

$$h\omega \ll k_B T : S(\omega) = 4Rk_B T$$

Over a range of frequencies:

$$\boxed{P(\Delta f) \propto k_B T \; \Delta f}$$

Nyquist-Johnson thermal noise (1928)



(Clay Turner, Wireless Solutions, 2007)

Detectability of scatter from ionospheric plasma

Finally,

$$P_{noise} \sim 2 \times 10^{-15} W$$

$$S/N \sim 5$$

Workable!

But you need a megawatt class transmitter and a huge antenna.

1950s: technology makes this possible (radio astronomy + construction = large antennas, military needs = high power transmitters)

Incoherent Scatter Concepts Are Older Than You Think

ACADÉMIE DES SCIENCES.

SÉANCE DU LUNDI 5 NOVEMBRE 1928.

PRÉSIDENCE DE M. MAURICE HAMY.

MÉMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

PHYSIQUE. — *Remarques sur la diffusion de la lumière et des ondes hertziennes par les électrons libres.* Note de M. CH. FABRY.

Un électron placé sur le trajet d'un faisceau de radiations électromagnétiques prend un mouvement vibratoire sous l'action du champ électrique de l'onde, et rayonne à son tour dans toutes les directions. Le phénomène ressemble à la *diffusion moléculaire*, qui peut être regardée comme due aux charges électriques contenues dans la molécule; mais les forces agissant sur ces charges ne sont pas les mêmes dans les deux cas, et les lois des deux phénomènes sont différentes. Tandis que la diffusion moléculaire est d'autant plus intense que la fréquence est plus élevée (loi en λ^{-4} de Lord Rayleigh), l'électron libre doit donner, pour une même intensité d'onde

Remarques sur la diffusion de la lumière et des ondes hertziennes par les electrons libres

C. Fabry
1928

Charles Fabry
1867-1945

$$\sigma = \frac{8\pi}{3}\frac{e^4}{m^2}$$

Electron scattering cross section (fundamental)

Incoherent Scatter Concepts Are Older Than You Think

Without worrying about noise:
- Rayleigh scattering $\sim\lambda^{-4}$ [why is the sky blue?]
- Incoherent scatter independent of wavelength
  [but it's weak]

Remarques sur la diffusion de la lumière et des ondes hertziennes par les electrons libres

C. Fabry
1928

Charles Fabry
1867-1945

Incoherent Scatter concept!

For luminous radiations whose wavelength is very small, there is no phase relation between the elementary waves sent out by the different electrons of even a small volume and it is the intensities which add up. Thus, if a certain volume contains a total number of electrons n, then the power that it scatters is that transmitted by an area S = nσ. With the degrees of ionization that can actually exist, the scattering of light by electrons is always very slight. That is why it plays no appreciable role in the production of light in the diurnal sky *.

First Incoherent Scatter Radar

- W. E. Gordon of Cornell is credited with the idea for ISR.
- *"Gordon (1958) has recently pointed out that scattering of radio waves from an ionized gas in thermal equilibrium may be detected by a powerful radar."* (Fejer, 1960)
- Gordon proposed the construction of the Arecibo Ionospheric Observatory for this very purpose (NOT for radio astronomy as the primary application)

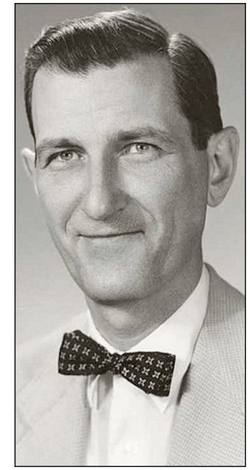
~40 megawatt-acres

- 1000' Diameter Spherical Reflector
  - 62 dB Gain
- 430 MHz line feed 500' above dish
- Gregorian feed
- Steerable by moving feed.

Proceedings of the
IRE, November 1958

Incoherent Scattering of Radio Waves by Free Electrons with Applications to Space Exploration by Radar*

W. E. GORDON†, MEMBER, IRE



Introduction

FREE electrons in an ionized medium scatter radio waves incoherently so weakly that the power scattered has previously not been seriously considered. The calculations that follow show that this incoherent scattering, while weak, is detectable with a powerful radar. The radar, with components each representing the best of the present state of the art, is capable of:

1) measuring electron density and electron temperature as a function of height and time at all levels in the earth's ionosphere and to heights of one or more earth's radii;
2) measuring auroral ionization;
3) detecting transient streams of charged particles coming from outer space; and
4) exploring the existence of a ring current.

* Original manuscript received by the IRE, June 11, 1958; revised manuscript received, August 25, 1958. The research reported in this paper was sponsored by Wright Air Dev. Ctr., Wright-Patterson Air Force Base, O., under Contract No. AF 33(616)-5547 with Cornell Univ.
† School of Elec. Eng., Cornell Univ., Ithaca, N. Y.

First Incoherent-Scatter Radar

- **K.L. Bowles [Cornell PhD 1955]**, Observations of vertical incidence scatter from the ionosphere at 41 Mc/sec. *Physical Review Letters* 1958:

"*The possibility that incoherent scattering from electrons in the ionosphere, vibrating independently, might be observed by radar techniques has apparently been considered by many workers although seldom seriously because of the enormous sensitivity required…*"

First Incoherent-Scatter Radar

…*Gordon (*W.E. Gordon from Cornell*) recalled this possibility to the writer* [spring 1958; D. T. Farley] *while remarking that he hoped soon to have a radar sensitive enough to observe electron scatter in addition to various astronomical objects…"*

Bowles executed the idea - hooked up a large transmitter to a dipole antenna array in Long Branch Ill., took a few measurements.

VOLUME 1, NUMBER 12 PHYSICAL REVIEW LETTERS DECEMBER 15, 1958

Table I. Parameters of radar equipment used.

Operating frequency	40.92 Mc/sec
Peak pulse power	(4 to 6) $\times 10^6$ watts
Pulse duration	(50 to 150) $\times 10^{-6}$ sec
Average power	4×10^4 watts maximum
Receiver bandwidth	10, 15, or 30 kc/sec
Antenna cross section	116 $\times$ 140 meters (1024 half-wave elements in phase above ground)
Antenna polarization	north-south
Calculated antenna gain	~35 decibels/isotropic

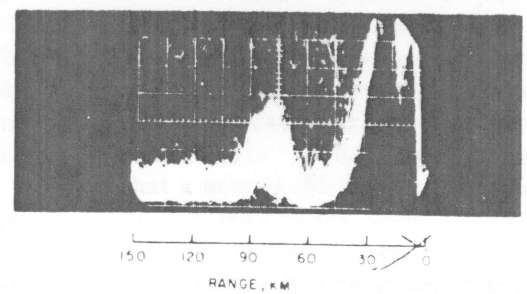


FIG. 2. Pulse width 50 μsec (~8 km); bandwidth 30 kc.

~6 week setup time

Oscilloscope + camera + ~4 sec exposure (10 dB integration)

Incoherent Scattering Detectability

Bowles' results found approximately the expected amount of power scattered from the electrons (scattering is proportional to charge to mass ratio - electrons scatter the energy).

BUT: his detection with a 20 megawatt-acre system at 41 MHz (high cosmic noise background; should be marginal) implies a spectral width 100x narrower than expected – almost as if the much heavier (and slower) ions were controlling the scattering spectral width.

In fact, they do.

Incoherent Scatter Radar

Particle-in-cell (PIC):

$$\frac{d\,\mathbf{v}_i}{d\,t} = \frac{q_i}{m_i}(\mathbf{E}(\mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(\mathbf{x}_i))$$

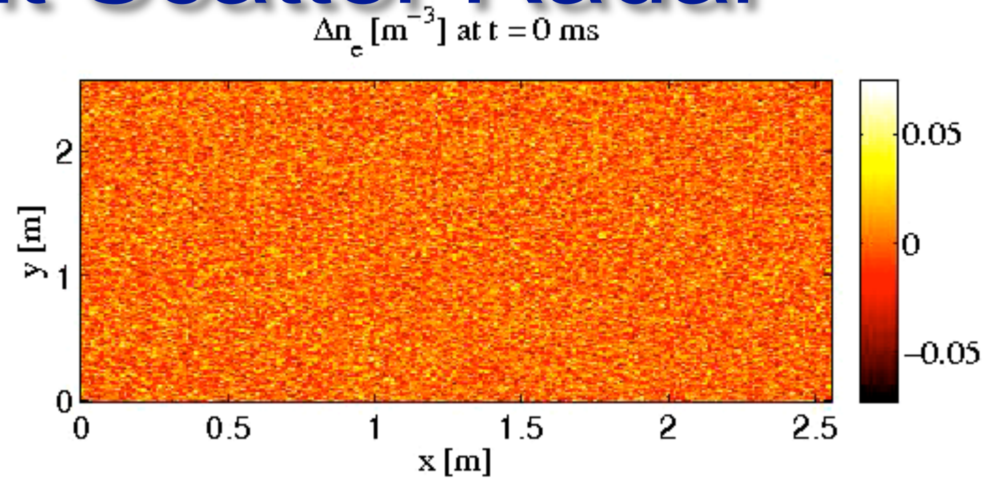
$$\nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}$$

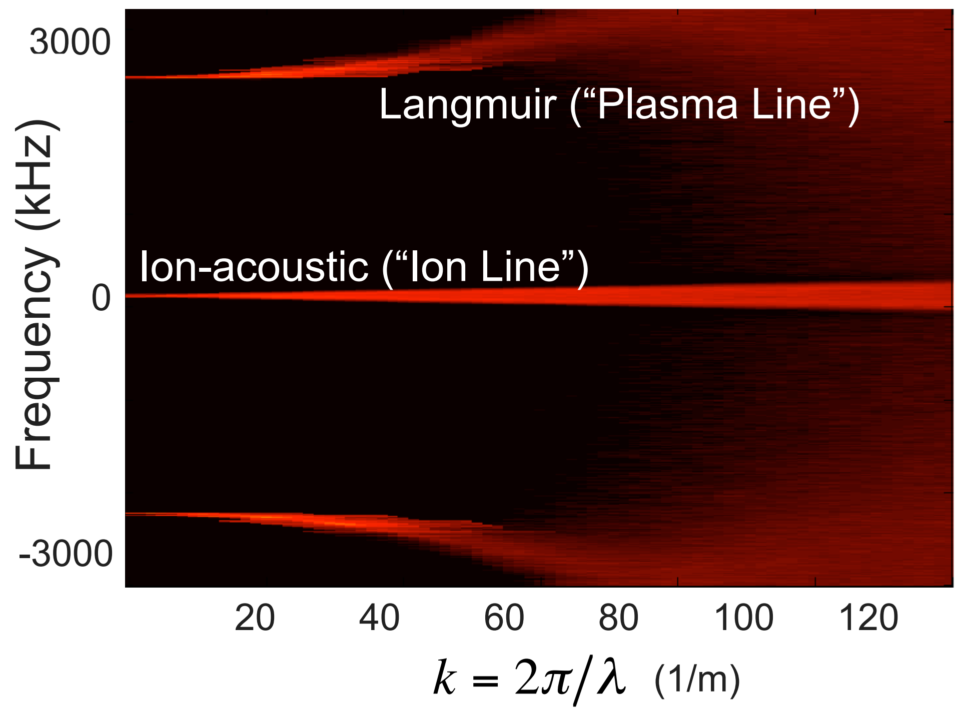
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

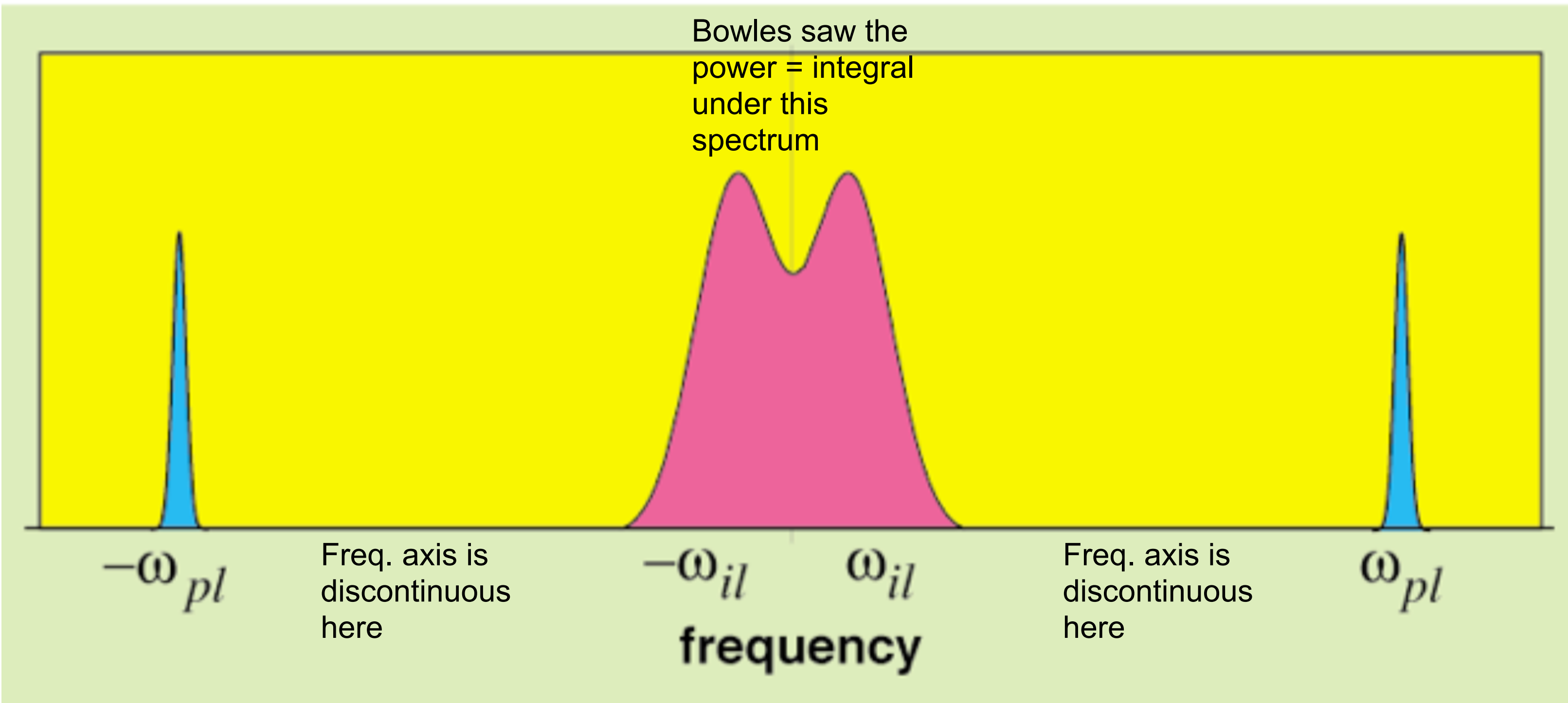
Simple rules yield complex behavior



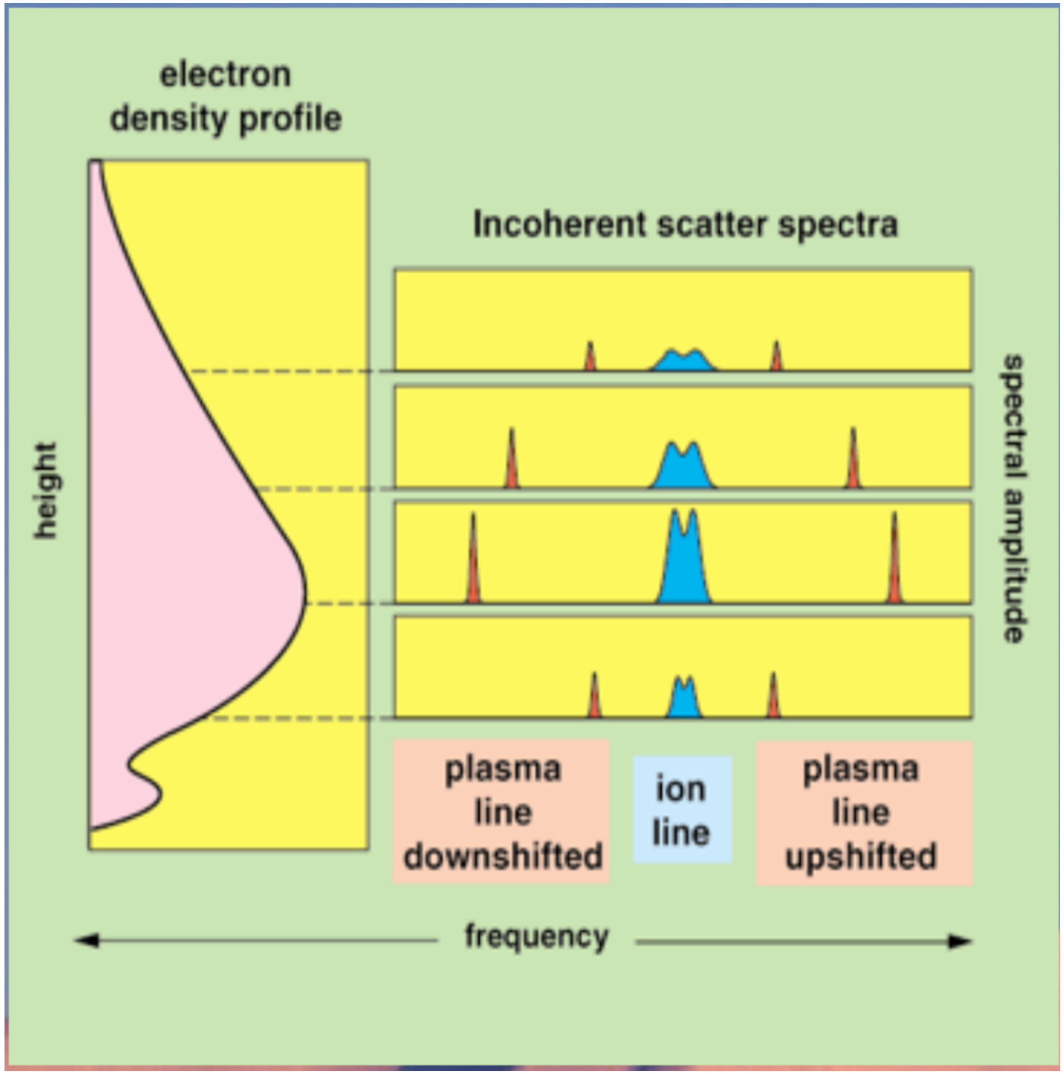


Diaz et al., RS 2008

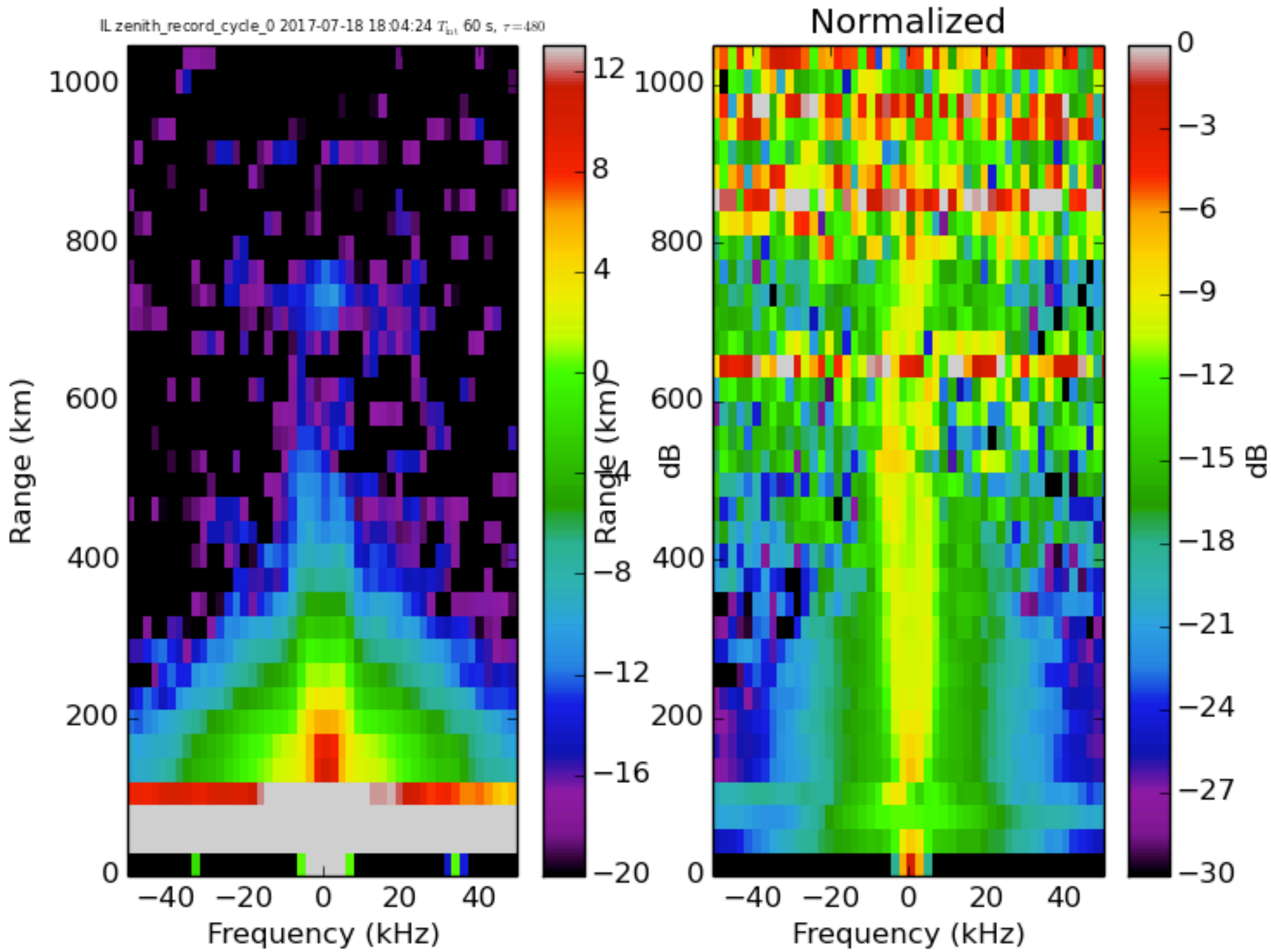
Incoherent Scatter Spectra as seen from a radar



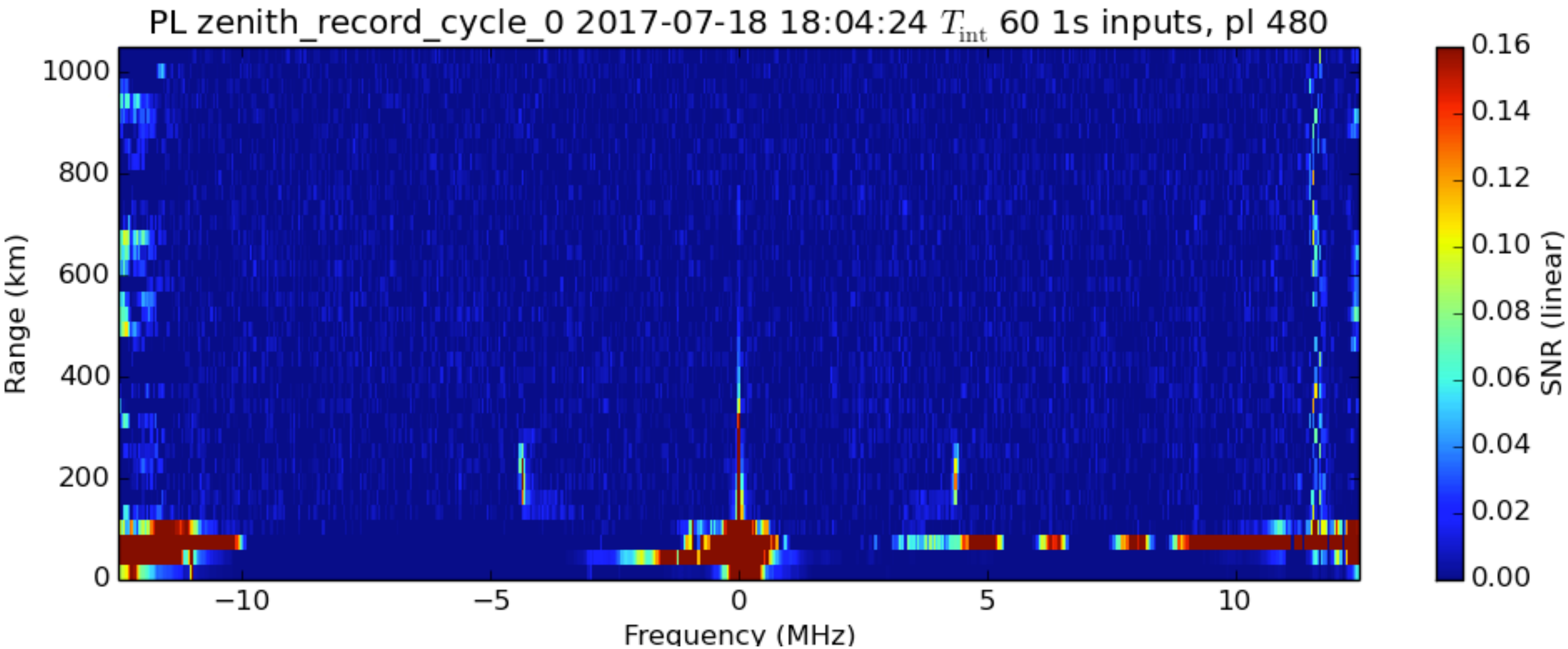
Incoherent Scatter Spectra as seen from a radar



Ion Line Spectra: Millstone Hill (60 sec integrations)

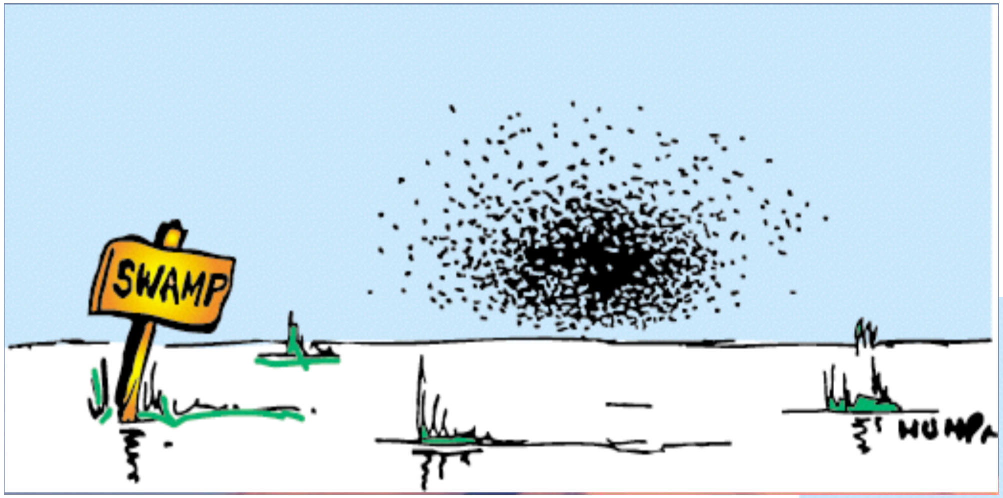


Plasma Line Spectra: Millstone Hill (60 sec integration)



Sanity Check

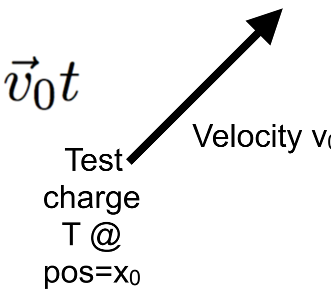
Theoretical Frameworks



Dressed Test Particle Theory

Assumptions:
- Particles move in straight lines
- No B field (electrostatics)

$$\vec{x}(t) = \vec{x}_0 + \vec{v}_0 t$$



Charge density of one test charge T: $$\rho_T(t) = q_T \; \delta(\vec{x} - \vec{x}_0 - \vec{v}_0 t)$$

Everything starts with Vlasov equation (no collisions here):

$$\frac{Df_s}{Dt} = 0$$

Vlasov Equation: Particles are Coupled by EM Interactions

Boltzmann equation, no collisions:

$$\frac{Df_s}{Dt} = 0 \quad \longrightarrow \quad \frac{\mathrm{d}\, f(\mathbf{r}, \mathbf{p}, t)}{\mathrm{d}\, t} = 0$$

(don't lose any particles in position space, momentum space, or time)

It's a PDE, so write out the complete derivative:

$$\frac{\partial f}{\partial t} + \frac{\mathrm{d}\,\mathbf{r}}{\mathrm{d}\,t} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

time change

space change

momentum change

Vlasov Equation: Particles are Coupled by EM Interactions

Look at this PDE again, where we are tracking particles:

$$\frac{\partial f}{\partial t} + \frac{\mathrm{d}\,\mathbf{r}}{\mathrm{d}\,t} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathrm{d}\,\mathbf{p}}{\mathrm{d}\,t} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0.$$

velocity

force (F = dp/dt)

Assume particles are not **collisionally** connected, but connected through common EM forces (Coulomb interactions).

$$\vec{F} = \frac{\partial \vec{p}}{\partial t} = \vec{E} + \vec{v} \times \vec{B}$$

Vlasov Equation: Particles are Coupled by EM Interactions

Vlasov recasts the Boltzmann concept using EM forcing:

electrons $$\frac{\partial f_e}{\partial t} + \mathbf{v}_e \cdot \nabla f_e - e\left(\mathbf{E} + \frac{\mathbf{v_e}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f_e}{\partial \mathbf{p}} = 0$$

ion species $$\frac{\partial f_i}{\partial t} + \mathbf{v}_i \cdot \nabla f_i + Z_i e\left(\mathbf{E} + \frac{\mathbf{v_i}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f_i}{\partial \mathbf{p}} = 0$$

Maxwell

$$\nabla \times \mathbf{B} = \frac{4\pi \mathbf{j}}{c} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Dressed Test Particle Theory

Simplify PDE for each species- **no collisions, no magnetic field (electrostatics)**:

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \frac{\partial f_s}{\partial \vec{x}} + \frac{\vec{F}}{m_s} \cdot \frac{\partial f_s}{\partial \vec{v}} = 0$$

$$\vec{F} = q_s \vec{E}$$

Electrostatic field: $$\vec{E} = -\nabla \phi = \frac{\partial \phi}{\partial \vec{x}}$$

Substitute: $$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \frac{\partial f_s}{\partial \vec{x}} - \frac{q_s}{m_s} \nabla \phi \cdot \frac{\partial f_s}{\partial \vec{v}} = 0$$

velocity forcing

Dressed Test Particle Theory

Take this and do a spatial Fourier transform:

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \frac{\partial f_s}{\partial \vec{x}} - \frac{q_s}{m_s} \nabla \phi \cdot \frac{\partial f_s}{\partial \vec{v}} = 0$$

velocity forcing

$$\frac{\partial f_{s\vec{k}}}{\partial t} + i \left(\vec{k} \cdot \vec{v} \right) f_{s\vec{k}} = \frac{i q_s}{m_s} \phi_{\vec{k}} \left(\vec{k} \cdot \frac{\partial f_{s\vec{k}}}{\partial \vec{v}} \right)$$

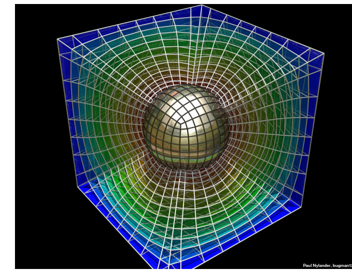
velocity forcing

Dressed Test Particle Theory

We need the potential now - employ Poisson's equation (electrostatics):

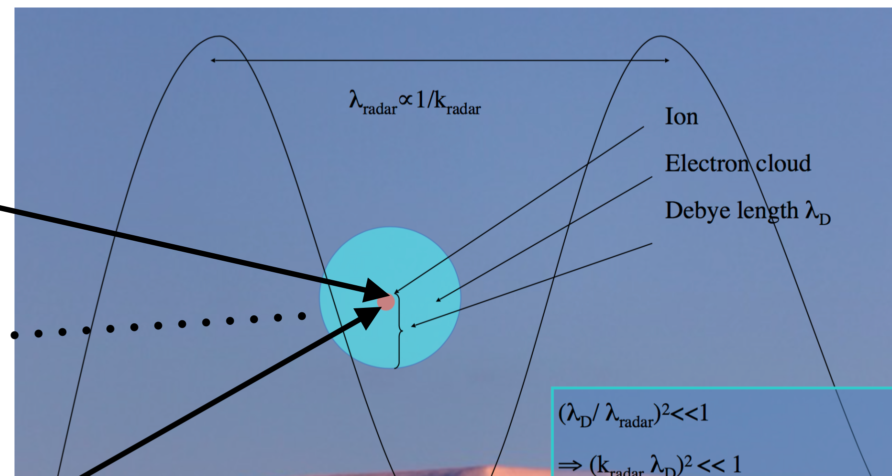
$$\nabla \cdot \vec{D} = \rho$$

or $$\nabla \cdot \vec{E} = \rho/\epsilon$$



Break the total charge up into two pieces:

1. Effect of each charge (test charge)
2. Effects of the perturbed charge in the plasma caused by the presence of each charge (we're in a plasma, so each charge partially attracts other charges - neutralizes some of the test charge)



$$\rho_T(t) = q_T \; \delta(\vec{x} - \vec{x}_0 - \vec{v}_0 t)$$

Dressed Test Particle Theory

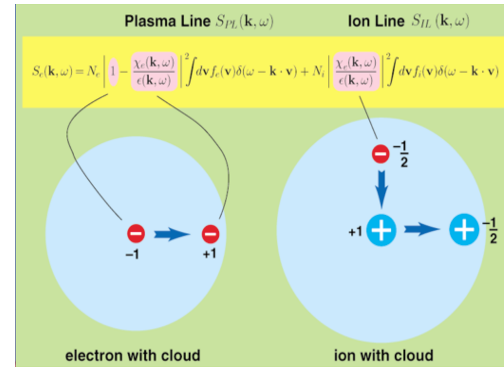
Poisson in this formulation

test charge

$$\nabla \cdot \vec{E} = -\nabla^2 \phi = \left(\sum_s \frac{q_s}{\epsilon_0} \int f_s \, d^3\vec{v} \right) + \frac{\rho_T}{\epsilon_0}$$

force

Perturbation in density of
background plasma
(partially neutralizing 'cloud')



Take this and do a spatial Fourier transform:

$$k^2 \phi_{\vec{k}} - \sum_s \frac{q_s}{\epsilon_0} \int f_s \, d^3\vec{v} = \frac{q_T}{\epsilon_0} \, e^{-i\vec{k} \cdot (\vec{x}_0 + \vec{v}_0 t)}$$

ES coupling
force

Perturbation in density of
background plasma
(partially neutralizing 'cloud')

test charge at (x_0, v_0) in pos/vel space

Dressed Test Particle Theory

Rewrite LHS in terms of a *relative* dielectric constant that takes into account the charge perturbation by the plasma 'cloud' as well as the test particle:

ES coupling force

Perturbation in density of background plasma (partially neutralizing 'cloud')

test charge

$$k^2 \phi_{\vec{k}} - \sum_s \frac{q_s}{\epsilon_0} \int f_s \, d^3\vec{v} = \frac{q_T}{\epsilon_0} \, e^{-i\vec{k}\cdot(\vec{x}_0 + \vec{v}_0 t)}$$

$$k^2 \phi_{\vec{k}} - \sum_s \frac{q_s}{\epsilon_0} \int f_{s\vec{k}} \, d^3\vec{v} \equiv k^2 \phi_{\vec{k}} \, \epsilon \left(\omega = \vec{k} \cdot \vec{v}_0, \vec{k}\right)$$

ES coupling force

Perturbation in density of background plasma (partially neutralizing 'cloud')

Relative dielectric constant

Dressed Test Particle Theory

So Poisson equation now takes the form

test charge

$$k^2 \phi_{\vec{k}} \, \epsilon \left(\omega = \vec{k} \cdot \vec{v}_0 , \vec{k} \right) = \frac{q_T}{\epsilon_0} \, e^{-i\vec{k} \cdot (\vec{x}_0 + \vec{v}_0 t)}$$

where

$$\epsilon = 1 + \sum_s \epsilon_s \qquad \epsilon_s = \frac{-1}{k^2 \phi_{\vec{k}}} \frac{q_s}{\epsilon_0} \int f_{s\vec{k}} \, d^3\vec{v}$$

free space dielectric response

collective effects (sum over species)

Perturbation in dielectric constant caused by partially neutralizing 'cloud' around species s (could be an ion or an electron)

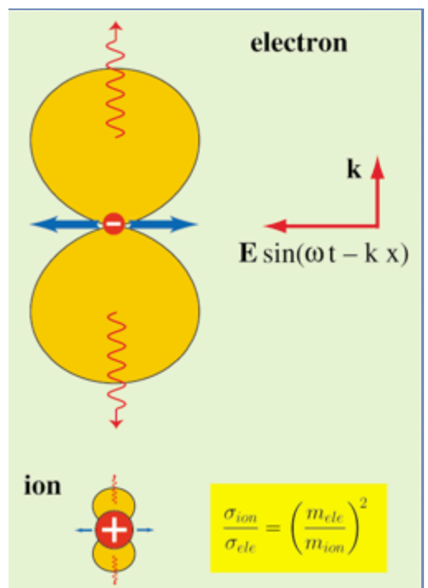
The physics of the medium is described by the dielectric constant (related to plasma conductivities)

Dressed Test Particle Theory

So now we can compute the potential and the dielectric constant (including perturbations from ‘clouds’ around test particles). These are collective effects and couple together **all** the particles.

Use this to find the electron density fluctuations in the medium (what we care about for scattering purposes). Now all sums are over individual particles.

Why only electrons?



Dressed Test Particle Theory

So now we can compute the potential and the dielectric constant (including perturbations from ‘clouds’ around test particles). These are collective effects and couple together **all** the particles.

Use this to find the electron density fluctuations in the medium (what we care about for scattering purposes). Now all sums are over individual particles.

$$\Delta N_e(\vec{k}, t) = \sum_{\text{ions, electrons}} [\text{electron clouds around test particles}]$$

$$+ \sum (\text{test electrons})$$

Dressed Test Particle Theory

But these are random fluctuations. We want the ensemble average of the particle time correlations - this has information.

some later time

$$\left\langle \Delta N_e(\vec{k}, t)\ \Delta N_e^*(\vec{k}, t') \right\rangle$$

by definition,
time avg of this = 0: $\langle E_s(t) \rangle = \langle \Delta N(\vec{r}, t) \rangle = 0$

Key insight: we assume the positions of each particle are uncorrelated - but the Coulomb force does influence their TIME behavior (**particle velocities are correlated by Coulomb coupling**). This takes out the positional information:

$$\left\langle e^{-i\vec{k}\cdot\vec{x}_{0m}}\ e^{i\vec{k}\cdot\vec{x}_{0n}} \right\rangle = 0 \text{ for } m \neq n$$

Expression becomes (several steps skipped here):

$$\left\langle \Delta N_e(\vec{k}, t)\ \Delta N_e^*(\vec{k}, t') \right\rangle = \sum_{\text{ions}} \left| \frac{\epsilon_e}{\epsilon} \right|^2 e^{-i\vec{k}\cdot\vec{v}_{0j}(t-t')} + \sum_{\text{electrons}} \left| 1 - \frac{\epsilon_e}{\epsilon} \right|^2 e^{-i\vec{k}\cdot\vec{v}_{0j}(t-t')}$$

collective effects: clouds around test particles

Dressed Test Particle Theory

Assuming 1D motion along k direction, power spectrum:

$$\left\langle \left| \Delta N_e(\vec{k},\omega) \right|^2 \right\rangle = \left[\sum_{\text{ion species}} \left| \frac{\epsilon_e(\omega,\vec{k})}{\epsilon(\omega,\vec{k})} \right|^2 f_{0s}(\omega/k) \right] + \left[\left| 1 - \frac{\epsilon_e(\omega,\vec{k})}{\epsilon(\omega,\vec{k})} \right|^2 f_{0e}(\omega/k) \right]$$

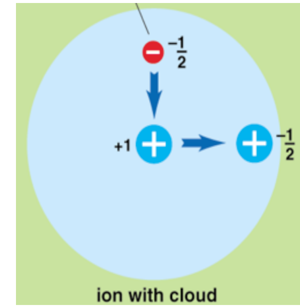
Proceed (not done here) with assuming a Maxwellian distribution and deriving the dielectric constants. **These must depend on plasma properties.**

We need this

In general, we find that

$$\epsilon(\omega,\vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Stop for a moment….

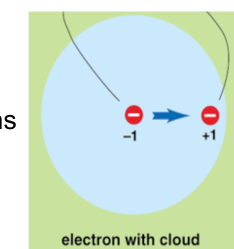


electron clouds around ions

$$\left\langle \left| \Delta N_e(\vec{k},\omega) \right|^2 \right\rangle = \left[\sum\nolimits_{\text{ion species}} \left| \frac{\epsilon_e(\omega,\vec{k})}{\epsilon(\omega,\vec{k})} \right|^2 f_{0s}(\omega/k) \right] + \left[\left| 1 - \frac{\epsilon_e(\omega,\vec{k})}{\epsilon(\omega,\vec{k})} \right|^2 f_{0e}(\omega/k) \right]$$

bare electrons

electron clouds around electrons



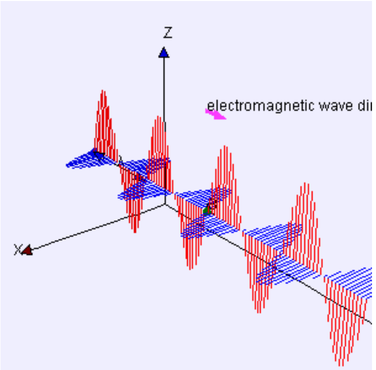
Dispersion relation: the concept

Key concept for wave behavior within a propagation medium.

Describes the relationship between SPATIAL frequency (wavelength) and TEMPORAL frequency.

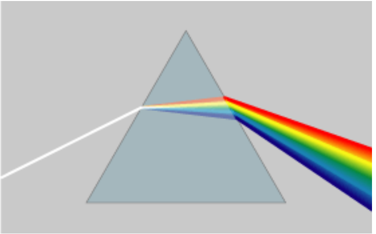
Some wave modes relate wavelength to frequency **linearly,** but waves in most media have **nonlinear** relation between wavelength and frequency.

Linear dispersion example:
EM radiation propagation through free space
(wavelength / velocity = c)



http://weelookang.blogspot.com/2011/10/ejs-open-source-propagation-of.html

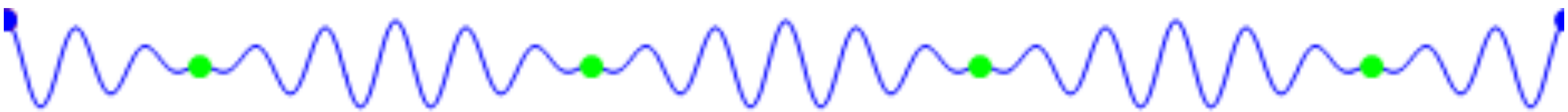
Nonlinear dispersion example:
splitting of light through a prism
(effective speed of light depends on wavelength due to glass' non-unity index of refraction)



Wikipedia CC-3.0

Dispersion relation: the concept

Linear dispersion in a transverse wave with 2 frequencies:
Note that phase (red) velocity = group (green) velocity



Nonlinear dispersion in a transverse wave with 2 frequencies:
Note that phase (red) velocity is **faster** here than group (green) velocity

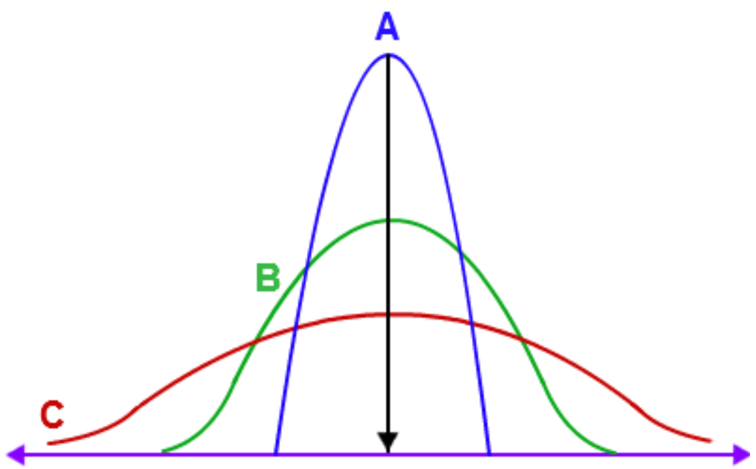
Dispersion relation: the concept

Simple linear case: uniform phase velocity

$$\omega(k) = c\ k$$

Most propagation speeds depend nonlinearly on the wavelength and/or frequency.

NB: for a **nonlinear** dispersion relation, the pulse will typically spread in either spatial frequency or temporal frequency as a function of time.



Example of pulse spreading spatially from time A to B to C.

http://www.mathcaptain.com/statistics/dispersion-statistics.html

Plasma dispersion relations

$$\epsilon(\omega, \vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

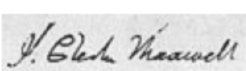
Dielectric constant of the medium

Insert plasma dispersion relation here

The physics of the medium is described by the dielectric constant (related to plasma conductivities)

Gauss' Law (electric field around charges)	$\nabla \cdot \mathbf{D} = \rho_f$	in free space: H = B D = E
Gauss' Law for magnetism (no magnetic monopoles)	$\nabla \cdot \mathbf{B} = 0$	
Faraday's Law (electric field around a changing magnetic field)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	
Ampere's Law (magnetic field circulation around electric charges)	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	

Maxwell's correction (displacement current)



(circles = places where dielectric constant shows up in Gauss, Ampere)

Important thermal plasma dispersion relations

$$\epsilon(\omega, \vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Dielectric constant of the medium

Insert plasma dispersion relation here

We need the full dispersion relation expression.

This is not a plasma waves course so we won't derive it, but the two most important modes are:

1) *Ion-acoustic fluctuations* [sound waves in plasma]

$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = V_s$$

(Note no gamma term for electrons since they are isothermal, but ions are slow and suffer 1D compressions so their gamma term = 3)

NB: ordinary acoustic waves: adiabatic compression / decompression of fluid particles.

Ion-acoustic fluctuations: restoring force = electromagnetic

Important thermal plasma dispersion relations

$$\epsilon(\omega, \vec{k}) = \text{function}\left(\omega^2/k^2\right)$$

Insert plasma dispersion relation here

We need the full dispersion relation expression.

This is not a plasma waves course so we won't derive it, but the two most important modes are:

2) *Langmuir oscillations* (Plasma oscillations):

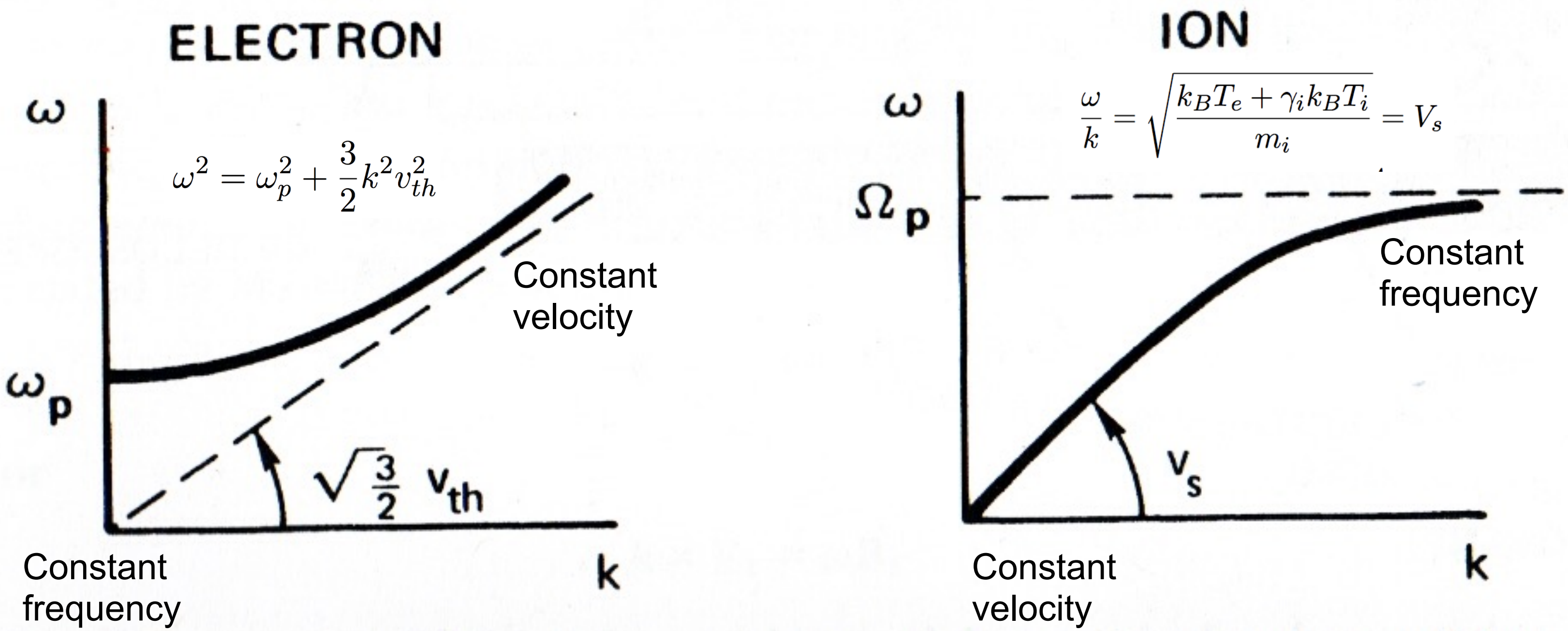
$$\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_{th}^2 \quad v_{th}^2 = 2k_B T_e/m_e$$

Akin to Brunt-Våisålå oscillations in fluid (parcel in presence of density gradient) - here, electrostatic field is restoring force, and electron pressure gradient transmits information

Historical aside: The Langmuir Mode

Irving Langmuir (1881 - 1957)

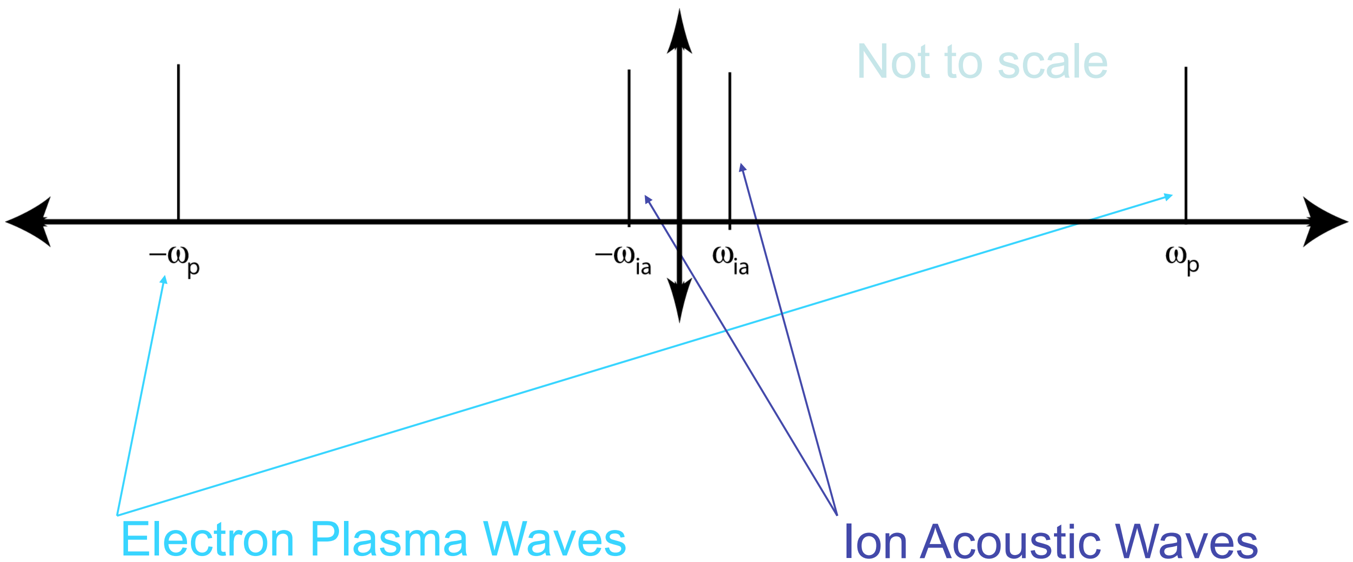
Electron and ion waves: Dispersion relations



Comparison of the dispersion curves for electron plasma waves and ion acoustic waves.

(Chen, Intro to Plasma Physics)

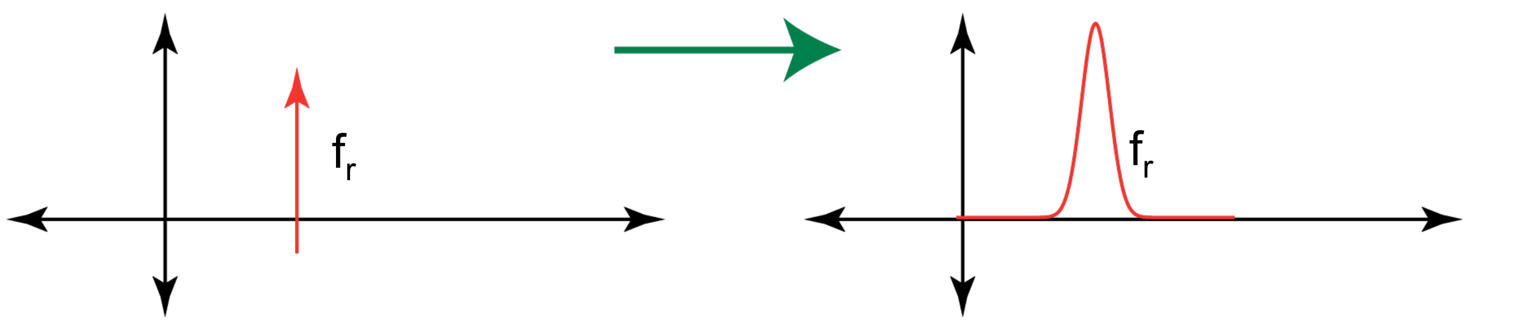
Wave Spectrum



Plasma parameters fluctuate with the waves (density, velocity, etc)

Damped resonance

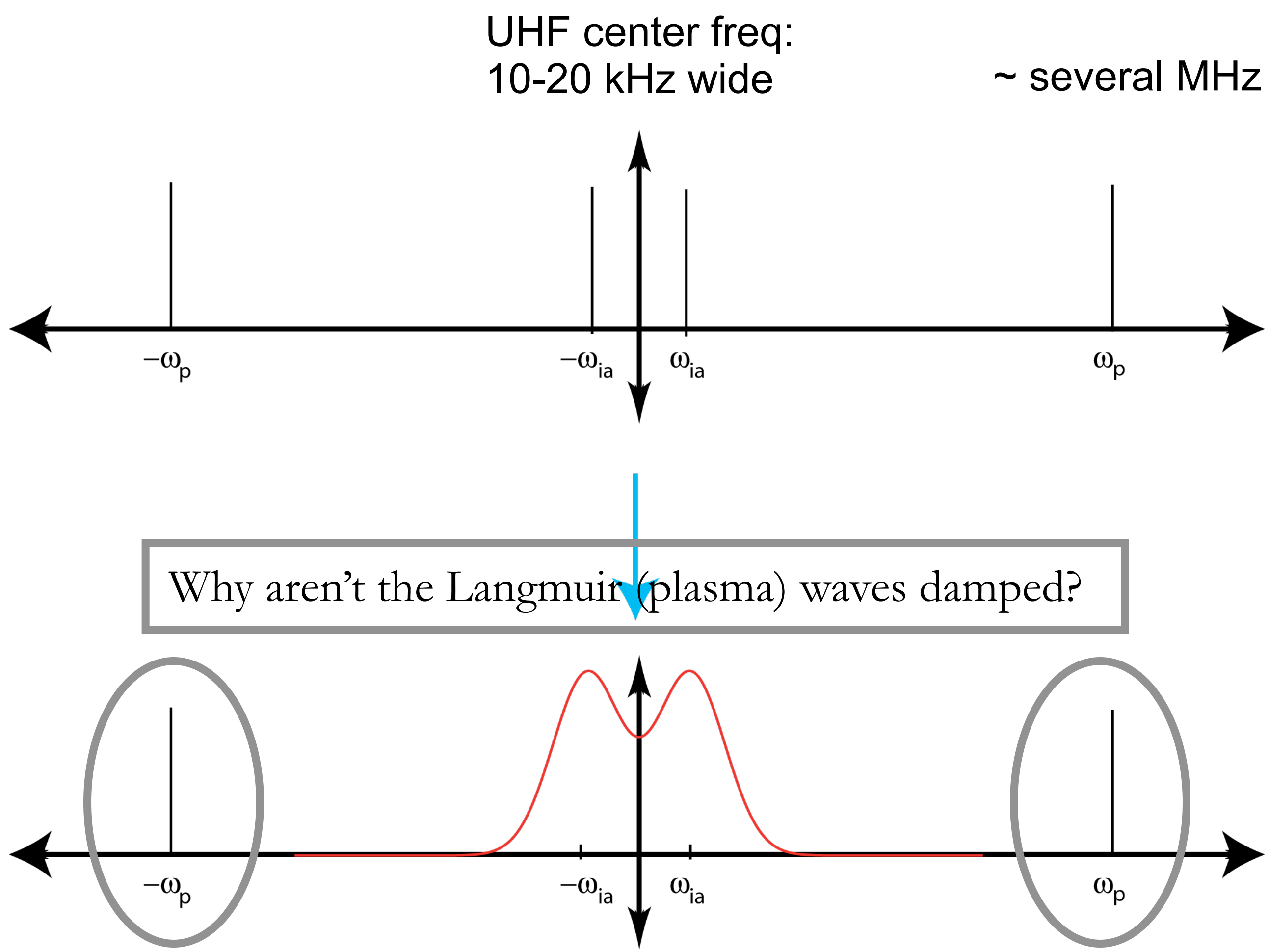
- Waves in a plasma are resonances.
- Damped resonances are not sharp
  - Example – Q of a resonant circuit.
- IS: Thermal ions have motions close to ion-acoustic speed (Landau damping – “surfing”; locked to I-A waves)



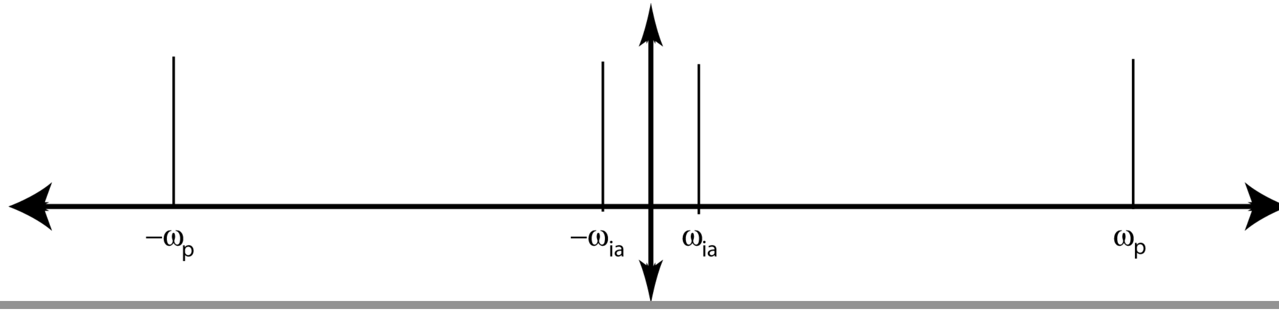
Resonance

Damped Resonance

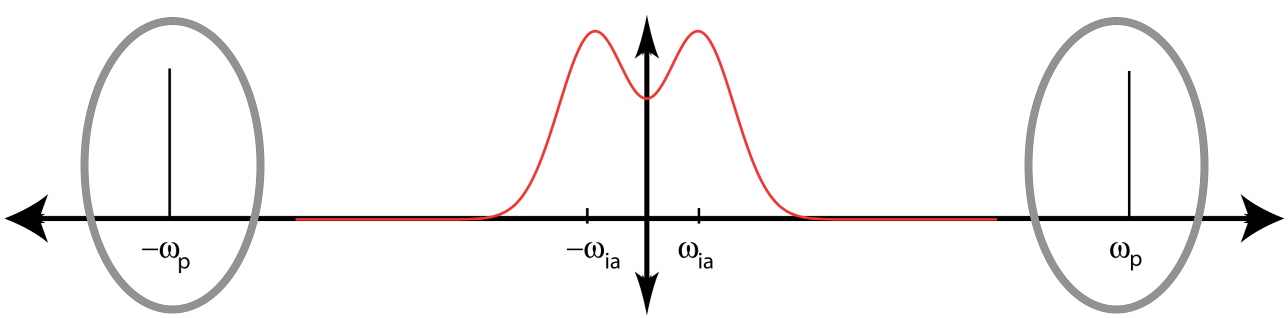
Wave Spectrum (ISR Spectrum)



Wave Spectrum (ISR Spectrum)

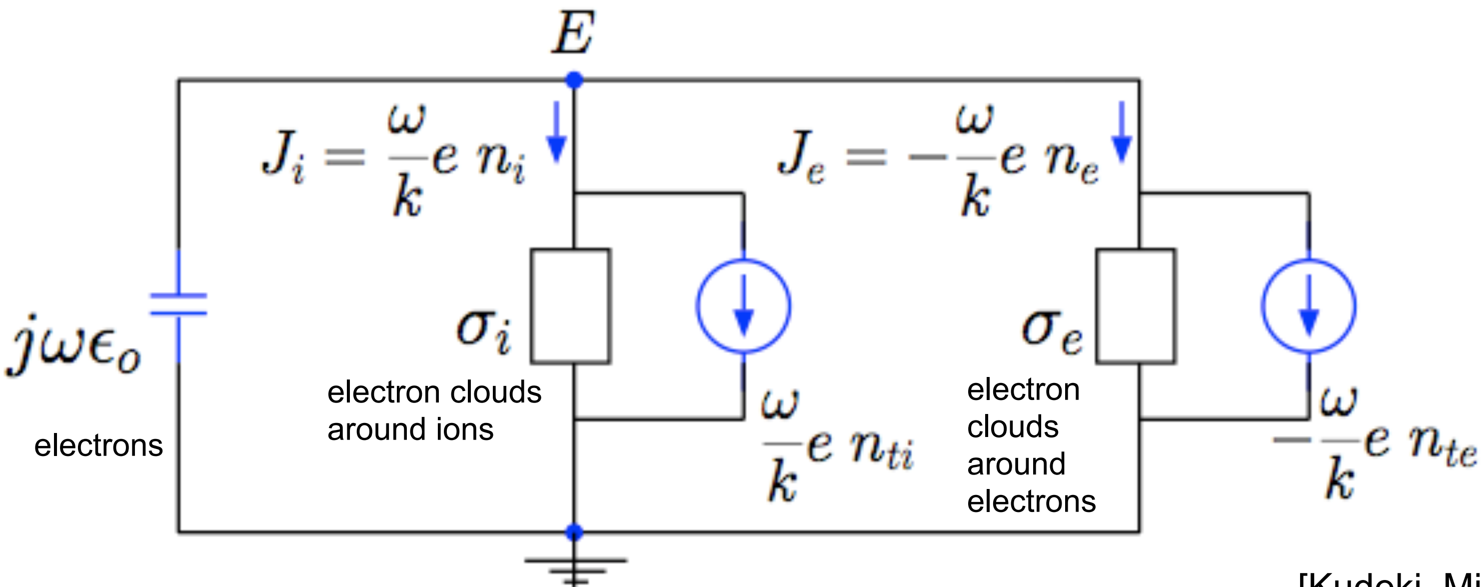


Why aren't the Langmuir (plasma) waves damped?
Electron thermal velocity ~ 125 km/s but plasma wave frequency ~ several MHz –
Not much interaction and not much damping.



Incoherent Scattering Model

$$\sigma_0(\omega_o + \omega)d\omega = N_0 r_e^2 \,\mathrm{Re}\left\{ \frac{y_e(y_i + jk^2\lambda_{de}^2)}{y_e + y_i + jk^2\lambda_{de}^2} \frac{d\omega}{\pi\omega} \right\}$$



[Kudeki, Milla]

Incoherent Scattering Model

$$\sigma_0(\omega_o + \omega)d\omega = N_0 r_e^2 \,\mathrm{Re}\left\{ \frac{y_e(y_i + jk^2\lambda_{de}^2)}{y_e + y_i + jk^2\lambda_{de}^2} \frac{d\omega}{\pi\omega} \right\}$$

- Short wavelength limit ($k^2\lambda_{de}^2 >> 1$): pure e^- scatter
- Long wavelength limit: RHS $\rightarrow y_e y_i/(y_e + y_i)$: damped ion-acoustic resonances
- Near plasma frequency: $y_e + y_i + jk^2\lambda_{de}^2 \rightarrow 0$: plasma lines

Incoherent Scattering Model

Spectral response can be evaluated using these frameworks for:

- Thermal inequality $T_e \neq T_i$: decreases Landau damping
- Ion-neutral collisions ν_{in}: narrows spectrum
- Background magnetic field B_0: makes electrons heavier

$$m_e \longrightarrow m_e^* = \frac{m_e}{\cos^2 \alpha}$$

Also, ion gyro-resonance (mass-dependent).

Incoherent Scattering Model

- Ion mixtures: $\frac{T_e}{T_i} y_i \rightarrow \sum_j \frac{T_e}{T_j} \frac{N_j}{N_0} y_j(m_j, T_j)$
- Unequal ion temperatures
- Particle drifts: $\omega \rightarrow \omega - \vec{k} \cdot \vec{v}_{de}$
- Plasma line measurements ($[e^-], T_e, v_\parallel$)
- Photoelectron heating, non-Maxwellian plasmas
- Faraday rotation effects (equator, low TX freq)

Things can get hairy. For example, magnetic field evaluation requires Gordeyev integral:

$$\int e^{j(\theta - j\phi)t - \frac{\sin^2\alpha}{\phi^2} \sin^2(\frac{\phi t}{2}) - \frac{t^2}{4} \cos^2\alpha} dt$$

Incoherent Scattering Model

(See IS Spectrum Java applet on “ISR Demonstration” page)

Incoherent Scatter Radar Remote Sensing: Summary

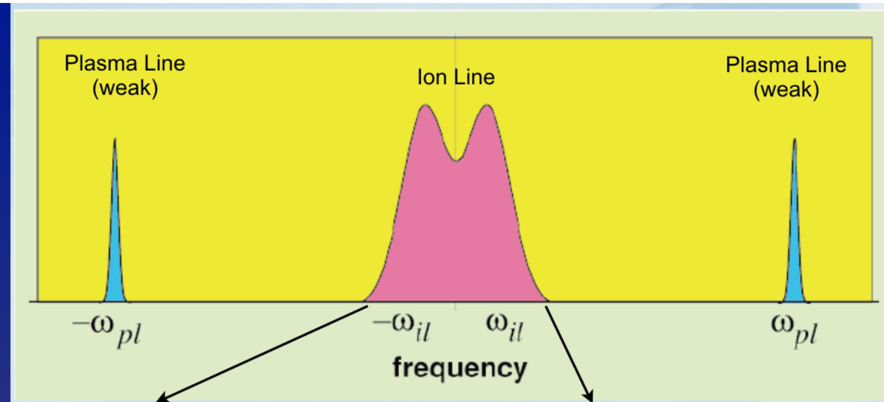
IS Radar Remote Sensing Capabilities

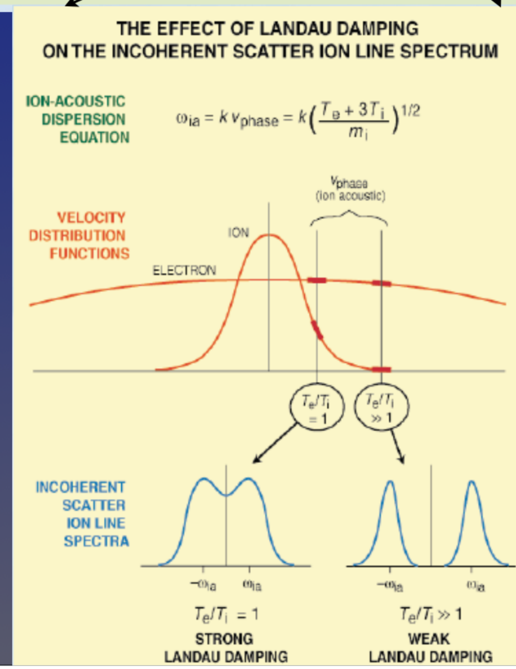
Parameters sensed (Full altitude profiles):

Basic
- Electron density
- Electron temperature
- Ion temperature
- Ion composition
- LOS Velocity

Derived
- Neutral winds
- Neutral temperature
- Vector velocity

More parameters possible





Example: "Ion Line" Sensitivity to Plasma Temperatures