Incoherent scatter theory II

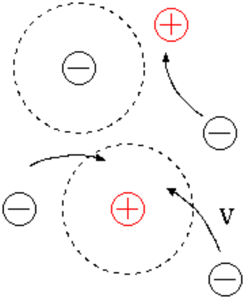
D. L. Hysell

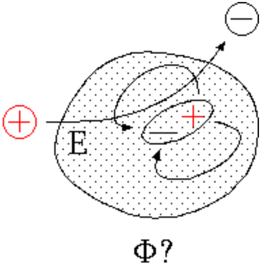
Earth and Atmospheric Sciences, Cornell University

July 17, 2015

introduction

- radar scatter arises from thermal fluctuations in electron density
- fluctuations are intrinsic (from discreteness of electrons) and induced (from Debye shielding of ions, other electrons)
- situation is analogous to a dielectric in which dipole moments are induced by the presence of free charge
- challenge is to compute fluctuations self-consistently
- use continuous formulation to study discrete particle behavior





Look out for characteristic functions, χ, ϵ, G.

intrinsic fluctuations

for collection of non-interacting particles ...

$$n(\mathbf{x}, t) = \sum_{i=1}^{N} \delta(\mathbf{x} - (\mathbf{x}_i + \mathbf{v}_i t))$$

$$n(\mathbf{k}, t) = \sum_{i=1}^{N} e^{-i\mathbf{k}\cdot(\mathbf{x}_i + \mathbf{v}_i t)}$$

with the autocorrelation function (ACF) ...

$$\begin{aligned}
\rho(\mathbf{k}, \tau) &= \langle n^*(\mathbf{k}, t) n(\mathbf{k}, t+\tau) \rangle \\
&= \left\langle \sum_i \sum_j e^{i\mathbf{k}\cdot(\mathbf{x}_i + \mathbf{v}_i t)} e^{-i\mathbf{k}\cdot(\mathbf{x}_j + \mathbf{v}_j (t+\tau))} \right\rangle \\
&= N \langle e^{-i\mathbf{k}\cdot\mathbf{v}\tau} \rangle
\end{aligned}$$

characteristic functions

given a Gaussian thermal distribution, find

$$\begin{aligned}\langle e^{-i\mathbf{k}\cdot\mathbf{v}\tau}\rangle &\equiv \int f_\circ(\mathbf{v}) e^{-i\mathbf{k}\cdot\mathbf{v}\tau} d^3v \\ &= e^{-\frac{1}{2}(kv_{\mathrm{th}}\tau)^2}\end{aligned}$$

with the corresponding spectrum:

$$\begin{aligned}\langle |n(\mathbf{k},\omega|^2\rangle &\propto \int_{-\infty}^{\infty} \rho(\mathbf{k},\tau) e^{i\omega\tau} d\tau \\ &= 2\Re \underbrace{\int_0^{\infty} \rho(\mathbf{k},\tau) e^{i\omega\tau} d\tau}_{G(\mathbf{k},\omega)} \\ &= n_\circ \frac{\sqrt{2\pi}}{kv_{\mathrm{th}}} e^{-\frac{1}{2}(\omega/kv_{\mathrm{th}})^2}\end{aligned}$$

i.e. incoherent addition

induced fluctuations

$$k^2\epsilon(\mathbf{k},\omega)\ \underbrace{\phi(\mathbf{k},\omega)}_{\substack{\text{self}-\text{consistent}\\ \text{potential}}} \quad = \quad \underbrace{e\left(n_i(\mathbf{k},\omega) - n_e(\mathbf{k},\omega)\right)}_{\substack{\text{intrinsic}\\ \text{fluctuations}}} \tag{1}$$

total electron density fluctuation:

$$\begin{aligned}\Delta n(\omega,\mathbf{k}) &= n_e(\mathbf{k},\omega) + n_\circ \int d^3v\, \delta f_1(\mathbf{k},\omega,\mathbf{v}) \\ &= n_e(\mathbf{k},\omega) + \frac{en_\circ}{m_e}\phi(\mathbf{k},\omega)\int d^3v\, \frac{\mathbf{k}\cdot\partial f_\circ/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}\end{aligned} \tag{2}$$

making use of the linearized Vlasov equation for unmagnetized, collisionless electrons to find f_1, i.e. $Df(\mathbf{x},\mathbf{v},t) = 0$:

$$(i\omega - i\mathbf{k}\cdot\mathbf{v})\, f_1 - ik(e/m_e)\phi(\mathbf{k},\omega)\partial f_\circ/\partial\mathbf{v} \quad = \quad 0$$

$$
\begin{aligned}
k^2\epsilon_\circ\phi(\mathbf{k},\omega) &= e(n_i - n_e) \\
&-\frac{e^2 n_\circ}{m_i}\phi\int d^3v\,\frac{\mathbf{k}\cdot\partial f_{\circ i}/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}} - \frac{e^2 n_\circ}{m_e}\phi\int d^3v\,\frac{\mathbf{k}\cdot\partial f_{\circ e}/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}
\end{aligned}
$$

or

$$
k^2\epsilon_\circ\phi(\mathbf{k},\omega)\underbrace{(1+\chi_i+\chi_e)}_{\epsilon(\mathbf{k},\omega)/\epsilon_\circ} = e(n_i - n_e)
$$

with

$$
\chi_{e,i} \equiv \frac{\omega_p^2}{k^2}\int d^3v\,\frac{\mathbf{k}\cdot\partial f_{\circ e,i}/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}
$$

Note that zeros of $\epsilon(\mathbf{k},\omega)$ are solutions to ES wave dispersion relation.

total MS electron density fluctuations

substituting ϕ from Eq. 1 along with the definitions of χ and ϵ into Eq. 2 gives:

$$\begin{aligned}\Delta n_e(\omega, \mathbf{k}) &= n_e(\mathbf{k},\omega) + \chi_e \frac{n_i(\mathbf{k},\omega) - n_e(\mathbf{k},\omega)}{\epsilon/\epsilon_\circ} \\ &= n_e(\mathbf{k},\omega) + \chi_e \frac{n_i(\mathbf{k},\omega) - n_e(\mathbf{k},\omega)}{1+\chi_e+\chi_i} \\ &= \frac{(1+\chi_i)n_e(\mathbf{k},\omega) + \chi_e n_i(\mathbf{k},\omega)}{1+\chi_e+\chi_i}\end{aligned}$$

$$\langle|\Delta n_e|^2\rangle = \frac{|1+\chi_i|^2}{|1+\chi_e+\chi_i|^2}\langle|n_e|^2\rangle + \frac{|\chi_e|^2}{|1+\chi_e+\chi_i|^2}\langle|n_i|^2\rangle$$

variance of the sum is the sum of the variance

relate the susceptibilities to the Gordeyev integral

$$G(\mathbf{k},\omega) = \int_0^\infty d\tau e^{i\omega\tau} \int f(\mathbf{v}) e^{-i\mathbf{k}\cdot\mathbf{v}\tau} d^3v$$

integrate by parts in velocity space, differentiate wrt frequency:

$$k^2 \frac{\partial G}{\partial \omega} = \int_0^\infty d\tau \int \mathbf{k} \cdot \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} e^{i(\omega - \mathbf{k}\cdot\mathbf{v})\tau} d^3v$$

perform τ integral:

$$k^2 \frac{\partial G}{\partial \omega} = -i \int d^3v \frac{\mathbf{k} \cdot \partial f/\partial \mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}} e^{i(\omega - \mathbf{k}\cdot\mathbf{v})\tau} \Big|_0^\infty$$

simplify

$$\frac{\partial G(\mathbf{k},\omega)}{\partial \omega} = i\chi/\omega_p^2$$

for the unmagnetized, collisionless species case, can go further (slide 5, integration by parts):

$$\begin{aligned} i\chi &= \omega_p^2 \int_0^\infty e^{-\frac{1}{2}(kv_{\mathrm{th}}\tau)^2} i\tau e^{i\omega\tau} d\tau \\ &= \frac{i}{k^2\lambda_d^2}\left[1 + i\omega G(\mathbf{k},\omega)\right] \end{aligned}$$

... and so there is an algebraic relationship between χ and G!

Incoherent scatter spectrum often written in terms of *admittance* functions:

$$1 + i\omega G(\mathbf{k},\omega) \quad = \quad iy(\mathbf{k},\omega)$$

leading to:

$$\langle|\Delta n_e|^2\rangle \quad = \quad n_\circ \frac{|(T_e/T_i)y_i - ik^2\lambda_{de}^2|^2 2\Re[y_e]/\omega + |y_e|^2 2\Re[y_i]/\omega}{|y_e + (T_e/T_i)y_i - ik^2\lambda_{de}^2|^2}$$

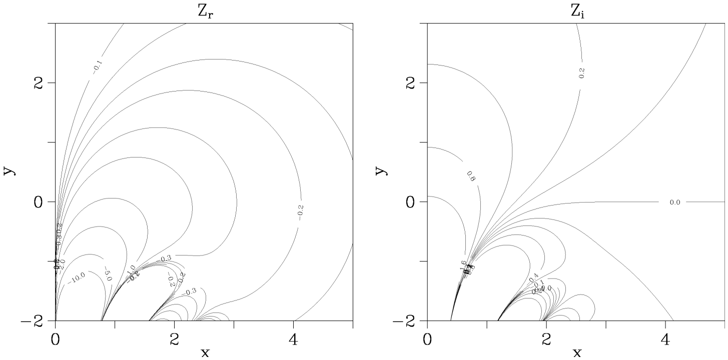
plasma dispersion function

$$\begin{aligned}\omega G &= \omega \int_0^\infty e^{-\frac{1}{2}k^2 v_{\rm th}^2 \tau^2} e^{i\omega\tau} d\tau \\ &= \int_0^\infty e^{-\frac{1}{4}x^2/\theta^2} e^{ix} dx \\ &= -i\theta Z(\theta)\end{aligned}$$

where the following definitions of the normalized frequency θ and plasma dispersion function Z apply

$$\begin{aligned}\theta &\equiv \frac{\omega/k}{\sqrt{2}v_{\rm th}} \\ Z(\theta) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2} dx}{x-\theta} \\ &= i\sqrt{\pi} e^{-\theta^2} (1 + \mathrm{erf}(i\theta))\end{aligned}$$

Z function



- $\langle e^{-i\mathbf{k}\cdot\mathbf{v}\tau}\rangle \to G(Z), y \to \langle |n|^2\rangle, \chi \to$ IS spectrum
- bulk drifts incorporated by taking $\omega_{e,i} \to \omega_{e,i} - \mathbf{k}\cdot\mathbf{v}_{e,i}$
- straightforward to incorporate collisions with neutral species by adding a BGK collision operator to the Vlasov equation
- also straightforward to include multiple ion species including negative ions:

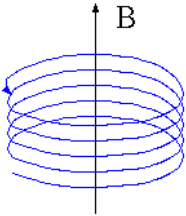
$$\frac{y_i}{T_i} \quad \to \quad \frac{1}{n_\circ e^2}\sum_j n_j q_j^2 \frac{y_j}{T_j}$$

- magnetic field effects also need to be considered, particularly at Jicamarca!

magnetic field effects

- calculations so far based on straight-line motion of non-interacting particles
- however, particles actually gyrate around geomagnetic field lines: $v_x \propto \cos(\Omega t)$, $v_y \propto \sin(\Omega t)$
- this can be regarded as straight-line motion in a rotating coordinate system!

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix} = \begin{pmatrix} & -\Omega & \\ \Omega & & \\ & & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$



$$\begin{pmatrix} r_1 \\ r_{-1} \\ r_\circ \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} & \\ 1/\sqrt{2} & -i/\sqrt{2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

B-field effects

$$
\begin{aligned}
\dot{v_\alpha} &= -i\alpha\Omega v_\alpha, \quad \alpha = \{1, -1, 0\} \\
v_\alpha(t) &= v_\alpha(0)e^{-i\alpha\Omega t} \\
v_\alpha(t-\tau) &= v_\alpha(t)e^{i\alpha\Omega\tau} \\
r_\alpha(t-\tau) - r_\alpha(t) &= \int_0^\tau v_\alpha(t-\tau)d\tau \\
&= v_\alpha(t)\frac{e^{i\alpha\Omega\tau}-1}{i\alpha\Omega} \\
&= v_\alpha g_\alpha(\tau)
\end{aligned}
$$

consequently, have a simple substitution ...

$$
\begin{aligned}
\left\langle e^{-i\mathbf{k}\cdot\mathbf{v}\tau} \right\rangle &\rightarrow \left\langle e^{-i\mathbf{a}\cdot\mathbf{v}} \right\rangle \\
\mathbf{a}\cdot\mathbf{v} &= k_{-\alpha} g_\alpha v_\alpha
\end{aligned}
$$

modified Gordeyev integral

for a Gaussian thermal distribution ...

$$\begin{aligned}\left\langle e^{-i\mathbf{a}\cdot\mathbf{v}}\right\rangle &= \int f_o(\mathbf{v}) e^{-i\mathbf{a}\cdot\mathbf{v}} d^3v \\ &= e^{-\frac{1}{2}v_t^2|a|^2}\end{aligned}$$

$$G(\mathbf{k},\omega) = \int_0^\infty dt \frac{1}{\sqrt{2}kv_{th}} e^{-i\theta t} e^{-\frac{1}{\phi^2}\sin^2\beta\sin^2(\frac{1}{2}\phi t) - \frac{1}{4}t^2\cos^2\beta}$$

$$\begin{aligned}\theta &\equiv \frac{\omega/k}{\sqrt{2}v_{th}} \\ \phi &\equiv \frac{\Omega/k}{\sqrt{2}v_{th}} \\ \mathbf{k}\cdot\mathbf{B} &= kB\cos\beta\end{aligned}$$

practical evaluation

note the following identity:

$$e^{-\frac{1}{\phi^2}\sin^2\beta\sin^2(\frac{1}{2}\phi t)} \quad = \quad e^{-\frac{1}{2\phi^2}\sin^2\beta}\sum_{n=-\infty}^{\infty} I_n\left(\frac{\sin^2\beta}{2\phi^2}\right)e^{i\phi n t}$$

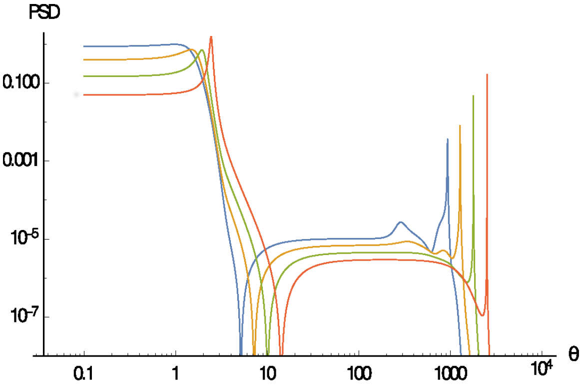
and so the final expression for for magnetized species is:

$$\omega G(\mathbf{k},\omega) \quad = \quad -i\frac{\theta}{\cos\beta}\sum_{n=-\infty}^{\infty} I_n\left(\frac{\sin^2\beta}{2\phi^2}\right)e^{-\frac{1}{2\phi^2}\sin^2\beta}Z\left(\frac{\theta-n\phi}{\cos\beta}\right)$$

... and so we're ready to calculate y_e, y_i, incoherent scatter spectrum (see slide 10)

- Magnetic field effects on ions are negligible except perhaps at VHF frequencies at propagation angles very close to perpendicular to **B**. This makes sense where the ion gyroradius is large compared to $\lambda/4\pi$, in which case the ions move in essentially straight lines.
- Except VHF and at very small magnetic aspect angles, magnetic field effect on electrons can be incorporated through an effective electron mass $m_e' \equiv m_e/\cos^2\beta$ which represents reduced electron mobility.
- At VHF and at small magnetic aspect angles, the effects of Coulomb collisions become significant. These effects presently cannot be treated analytically. Instead, a numerical estimate of y_e has been formulated based on monte-carlo simulations of electron trajectories. This work is ongoing.

example spectra



ISR spectrum for and oxygen plasma with $T_e/T_i = (1, 2, 4, 8)$ and for Arecibo conditions.