

Incoherent Scatter Theory: A Little Deeper Look

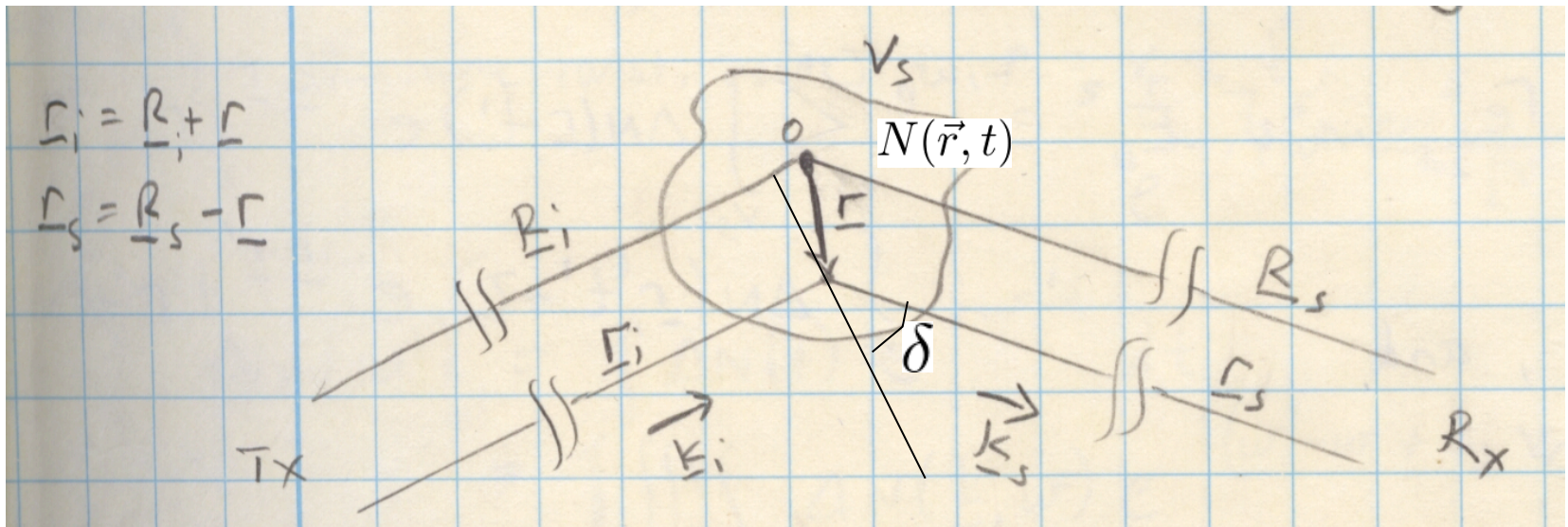
P. J. Erickson
ISR School 2014
Arecibo



Single Electron Scattering Model

Incident EM wave accelerates each charged particle it encounters. These then re-radiate an EM wave (as Hertzian dipoles).

For a single electron located at $r = 0$, we need the scattered field at a distance r_s .



Single Electron Scattering Model

Incident EM wave accelerates each charged particle it encounters.
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For a single electron located at $r = 0$, the scattered field at a distance r_s :

$$\text{scattered field } \left| \vec{E}_s(\vec{r}_s, t) \right| = \frac{e^2 \mu_0 \sin \delta}{4\pi r_s m_e} \left| \vec{E}_i(0, t') \right| \quad \text{Incident field}$$

$$= \frac{r_e}{r_s} \sin \delta \left| \vec{E}_i(0, t') \right|$$

$$r_e = \frac{e^2 \mu_0}{4\pi m_e} \quad \text{Classical electron radius}$$

$$t' = t - \frac{r_s}{c} \quad \text{Delayed time}$$

$$\sin \delta \quad \text{Scattering angle}$$

Scattering Model

Assume a volume filled with electron scatterers whose density is represented in space and time by

$$N(\vec{r}, t)$$

Illuminating this volume with an incident field from a transmitter location means that each electron contributes to the resulting scattered field, using *Born approximation* (each scatter is weak and does not affect others).

With geometrical considerations, scattered field at receiver location is now:

$$E_s(t) = r_e \sin \delta E_0 e^{j\omega_0 t''} \int_{V_s} \frac{1}{r_s} N(\vec{r}, t') e^{-j(\vec{k}_i - \vec{k}_s) \cdot \vec{r}} d^3 \vec{r}$$

$$t'' = t' - \frac{r_i}{c}$$

Delayed time (TX to RX)

$$t' = t - \frac{r_s}{c}$$

Delayed time (volume to RX)

Scattering Model

Assume densities have random spatial and temporal fluctuations about a background:

$$N(\vec{r}, t) \rightarrow N_0 + \Delta N(\vec{r}, t)$$

Further, assume backscatter (i.e. monostatic radar):

$$\vec{k} = 2\vec{k}_i \quad r_i \equiv r_s = R \quad \sin \delta \rightarrow 1$$

Then, scattered field reduces to:

$$E_s(t) \rightarrow \frac{r_e}{R} E_0 e^{j\omega_0 t''} \underbrace{\int_{V_s} \Delta N(\vec{r}, t') e^{-j\vec{k} \cdot \vec{r}} d^3 \vec{r}}_{\equiv \Delta N(\vec{k}, t')}$$

Maxwell's Equations



J. C. Maxwell
1831 - 1879

Governs propagation of electromagnetic waves ("action at a distance"), relation between electric and magnetic field and motions of charges

Foundation of classical electromagnetic theory

Gauss' Law (electric field around charges)

$$\nabla \cdot \mathbf{D} = \rho_f$$

in free space:
 $\mathbf{H} = \mathbf{B}$ $\mathbf{D} = \mathbf{E}$

Gauss' Law for magnetism (no magnetic monopoles)

$$\nabla \cdot \mathbf{B} = 0$$

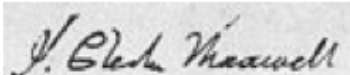
Faraday's Law (electric field around a changing magnetic field)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Ampere's Law (magnetic field circulation around electric charges)

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

↑
Maxwell's correction
(displacement current)



Scattering Model

Plasmas (ionosphere) are thermal gases and $\Delta N(\vec{r}, t)$ is a Gaussian random variable, so the Central Limit Theorem applies:

statistical average \longrightarrow $\langle E_s(t) \rangle = \langle \Delta N(\vec{r}, t) \rangle = 0$

It's much more useful to look at second order products – in other words, examine temporal correlations in the scattered field:

$$\langle E_s(t) E_s^*(t + \tau) \rangle \propto e^{-j\omega_0\tau} \langle \Delta N(\vec{k}, t) \Delta N^*(\vec{k}, t + \tau) \rangle$$

Useful things to measure can now be defined.

Scattering: Measurable Quantities

Defining $C_s = \frac{r_e^2 E_0^2 \sin^2 \delta}{R^2} V_s$, then

Total scattered power

$$\langle |E_s(t)|^2 \rangle = C_s \langle |\Delta N(\vec{k})|^2 \rangle$$

and Autocorrelation function (ACF):

$$\langle E_s(t) E_s^*(t + \tau) \rangle = C_s e^{-j\omega_0 \tau} \langle \Delta N(\vec{k}, t) \Delta N^*(\vec{k}, t + \tau) \rangle$$

or Power Spectrum:

$$\langle |E_s(\omega_0 + \omega)|^2 \rangle \propto C_s \langle |\Delta N(\vec{k}, \omega)|^2 \rangle$$

Incoherent Scattering Model: Summary

Radar filters in k space:

$$\Delta N(\vec{r}, t) \rightarrow \Delta N(\vec{k}_r, t)$$

$$\Delta N(\vec{k}_r, t) \propto E_s(t)$$

Form ACF of $E_s(t)$ for each range, average, transform:

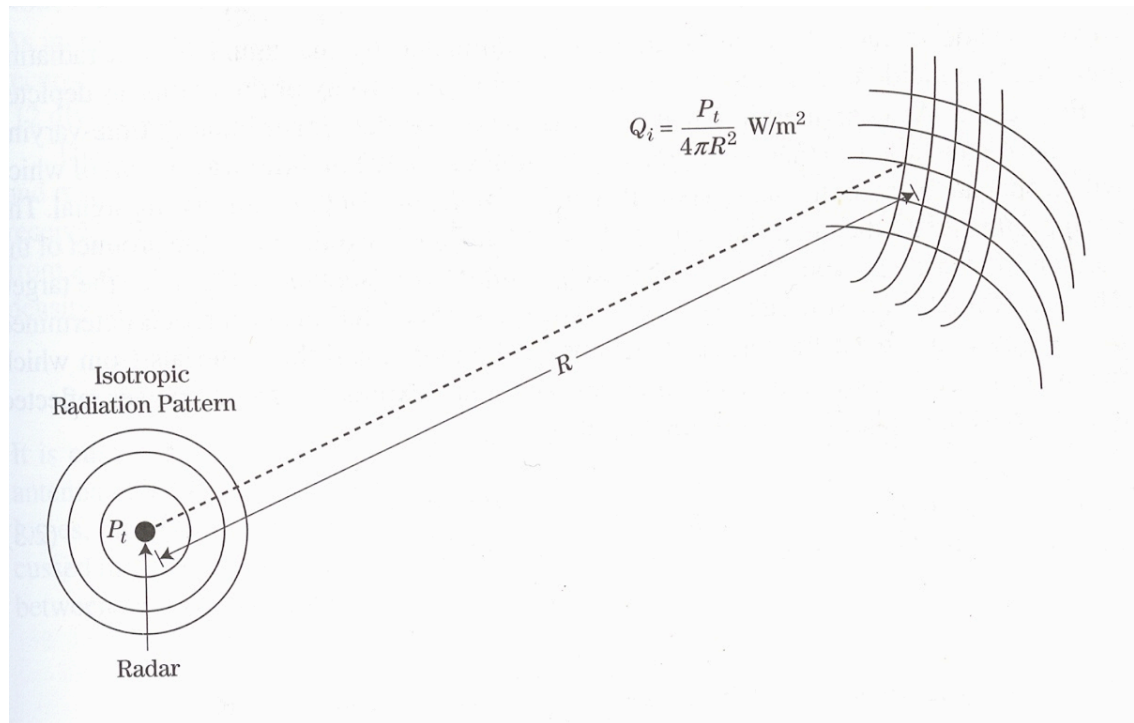
$$\langle E_s(t) E_s^*(t + \tau) \rangle \rightarrow \left\langle \left| \Delta N(\vec{k}, w) \right|^2 \right\rangle$$

Interpret latter in terms of the medium parameters.

The Radar Equation: Monostatic Version

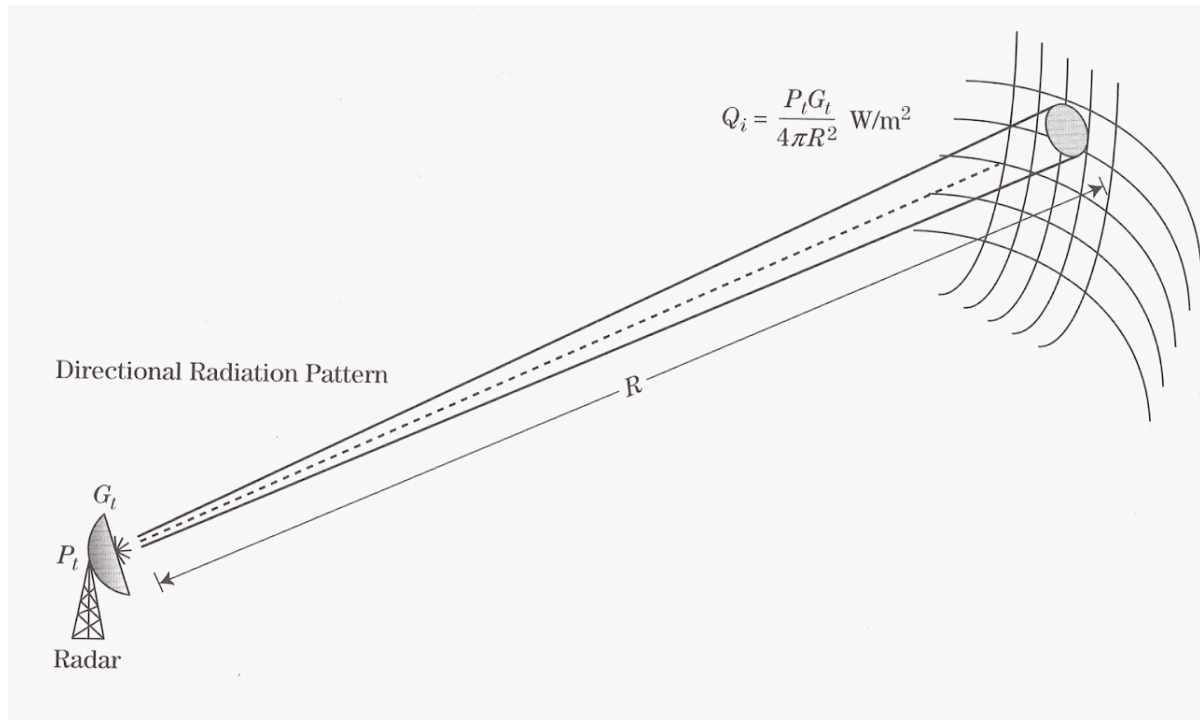
Power density at range R (isotropic):

$$\frac{P_t}{4\pi R^2}$$



The Radar Equation: Monostatic Version

Power density at range R (directional): $\frac{P_t G}{4\pi R^2}$

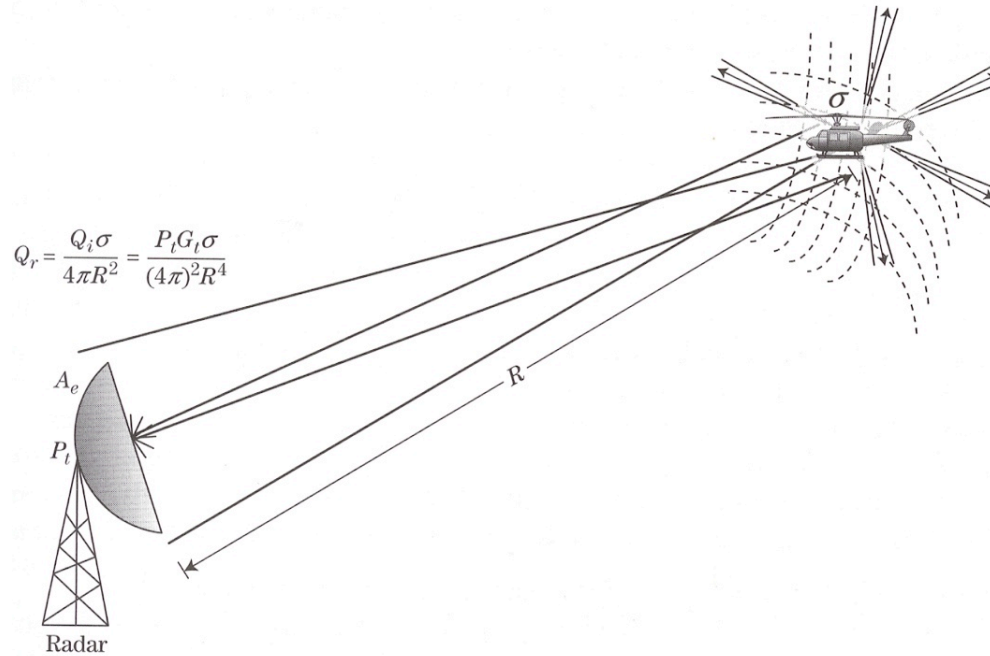


The Radar Equation: Monostatic Version

Radar cross section
(m²)

Reradiated power density at Rx:

$$\frac{P_t G}{4\pi R^2} \cdot \frac{\sigma}{4\pi R^2}$$



The Radar Equation: Monostatic Version

Total received power:
$$P_r = \frac{P_t G}{4\pi R^2} \cdot \frac{\sigma}{4\pi R^2} \cdot A_e = \frac{P_t G A_e \sigma}{(4\pi)^2 R^4}$$

Use gain/area relation -

The Radar Equation:

$$P_r = P_t \frac{\rho_a^2 A^2}{4\pi \lambda^2 R^4} \sigma$$

Maximum range form:

$$R_{max} = \left[\frac{P_t}{S_{min}} \frac{\rho_a^2 A^2}{4\pi \lambda^2} \sigma \right]^{\frac{1}{4}}$$

Hard vs Soft Radar Targets

Generalize radar equation for one or more scatterers, distributed over a volume:

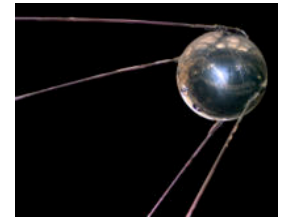
$$P_r = \int P_t \frac{\rho_a^2 A^2}{4\pi\lambda^2 R^4} \sigma(\vec{x}) dV_s$$

First case: single scatterer (“hard target”) at single point in space:

$$\int \sigma(\vec{x}) dV_s = \sigma_{target} \equiv \sigma$$

Hard target radar equation:

$$P_r = P_t \frac{\rho_a^2 A^2}{4\pi\lambda^2 R^4} \sigma$$

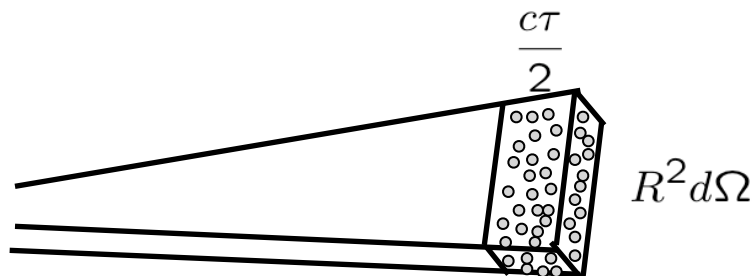


Sputnik 1 (1957-10-04)

Distributed Targets

$$\int \sigma(\vec{x}) dV_s = \int_0^{2\pi} \int_0^\pi \sigma(\vec{x}) \frac{c\tau}{2} R^2 d\Omega$$

$$\int \sigma(\vec{x}) dV_s = \frac{c\tau}{2} \int_0^{2\pi} \int_0^\pi \sigma(\vec{x}) R^2 \sin \theta d\theta d\phi$$



Assume volume is filled
with identical, isotropic
scatters

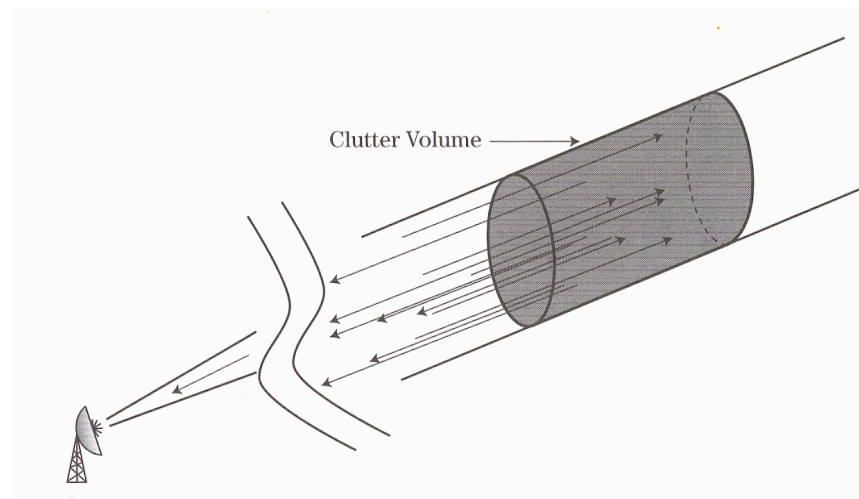
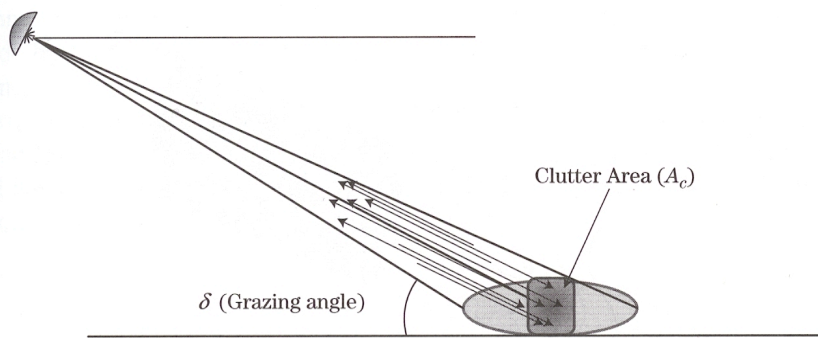
$$\int \sigma(\vec{x}) dV_s = \frac{c\tau}{2} R^2 \sigma$$

Distributed Scatterers

$$P_r = P_t \frac{\rho_a^2 A^2}{4\pi\lambda^2 R^4} \sigma \frac{c\tau}{2} R^2$$

The “soft target” Radar Equation

$$P_r = P_t \frac{c\rho_a^2 A^2 \tau}{8\pi\lambda^2 R^2} \sigma$$



Remote sensing a plasma: The experimental (radar) view

Suppose we transmit a wave towards a plasma and measure the scattered wave:

$$P_{rec} = (P_{inc}) A_{scat} \left(\frac{A_{rec}}{4\pi R^2} \right)$$

$$A_{scat} = \sigma_{radar} V_s \quad (\text{ionosphere is a beam filling target})$$

$$\sigma_{radar} = 4\pi \sigma_{total} \quad (\text{Solid angle})$$

$$\left(\frac{P_{rec}}{P_{inc}} \right) \left(\frac{4\pi R^2}{A_{rec}} \right) \left(\frac{1}{V_s} \right) = 4\pi r_e^2 \sin^2 \delta \langle |\Delta N(k)|^2 \rangle$$

Measurable experimentally

(1 for backscatter)

Physics info is here!

Radar cross-section of ionospheric plasma

Assume a beam filling plasma at F region altitudes (300 km) with very high electron density (1E12 electrons per m³ - BEST CASE):

Classical electron scattering cross-section $\sigma_e = 10^{-28} m^2 / e^-$

Assume a pulse length of 10 km.

Assume a cross-beam width of 1 km (~ Arecibo).

Total cross section is then (10 km x 1 km x 1 km x 1E12 m⁻³ x 1E-28 m⁻²/e⁻):

$$\sigma_{tot} \sim 10^{-6} m^2$$

-60 dBsm! Are we going to be able to do this at all?

NB: Born approximation is very valid, since total amount of scattered power in the volume ~ 1E-12. So we can make full range profiles if we can detect the scatter.

Detectability of scatter from ionospheric plasma

For fraction of scattered power actually received, assume isotropic scatter and a BIG 100 m class antenna:

$$f_{rec} = \frac{A_{rec}}{4\pi R^2} \sim \frac{10^4 m}{4(300 \times 10^3 m)^2}$$

About -80 dB (1E-8): not much. So:

$$\frac{P_{rec}}{P_{tx}} \sim 10^{-20}$$

So a radar with 1 MW transmitted signal receives 10 femtowatts of incoherently scattered power from free electrons in the ionosphere.

REALLY not very much.

Detectability of scatter from ionospheric plasma

What matters, though, is the signal to noise ratio:

$$P_{noise} = (k_B T_{eff}) (BW) \quad (\text{derived later})$$

Typical effective noise temperatures ~ 100 to 200 K at UHF frequencies (430 MHz, say).

Assume the bandwidth is set by thermal electron motions in a Boltzmann sense:

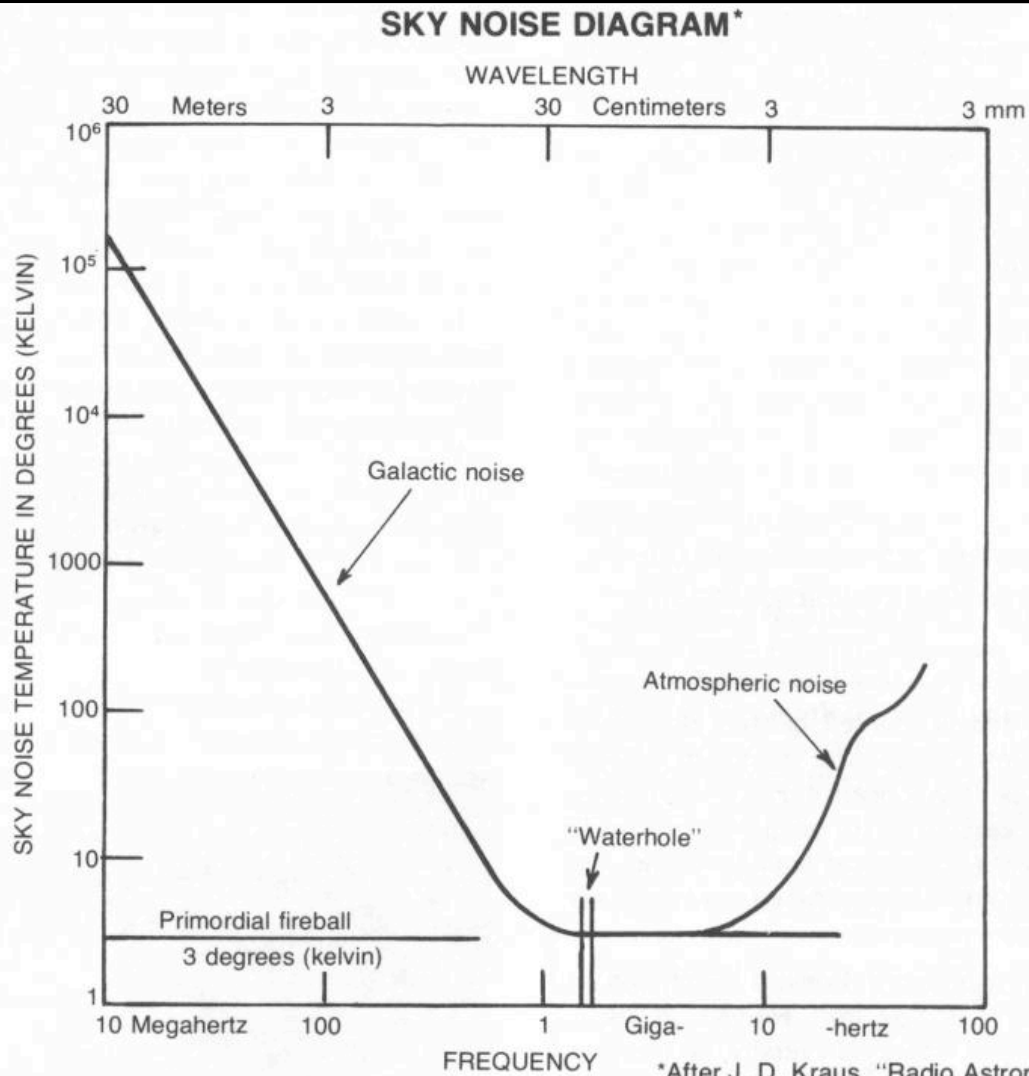
$$3k_B T_e \sim m_e v_{e,th}^2$$

$$v_{e,th} \sim \sqrt{\frac{3k_B T_e}{m_e}} \sim 2 \times 10^5 \text{ m/s}$$

$$BW \sim (v_{e,th}) (2)(2) \left(\frac{f_{tx}}{c}\right) \sim 10^6 \text{ Hz}$$

(2s are for up/down, backscatter)

Sky Noise: The Universe Is Also Transmitting



Detectability of scatter from ionospheric plasma

Finally,

$$P_{noise} \sim 2 \times 10^{-15} W$$

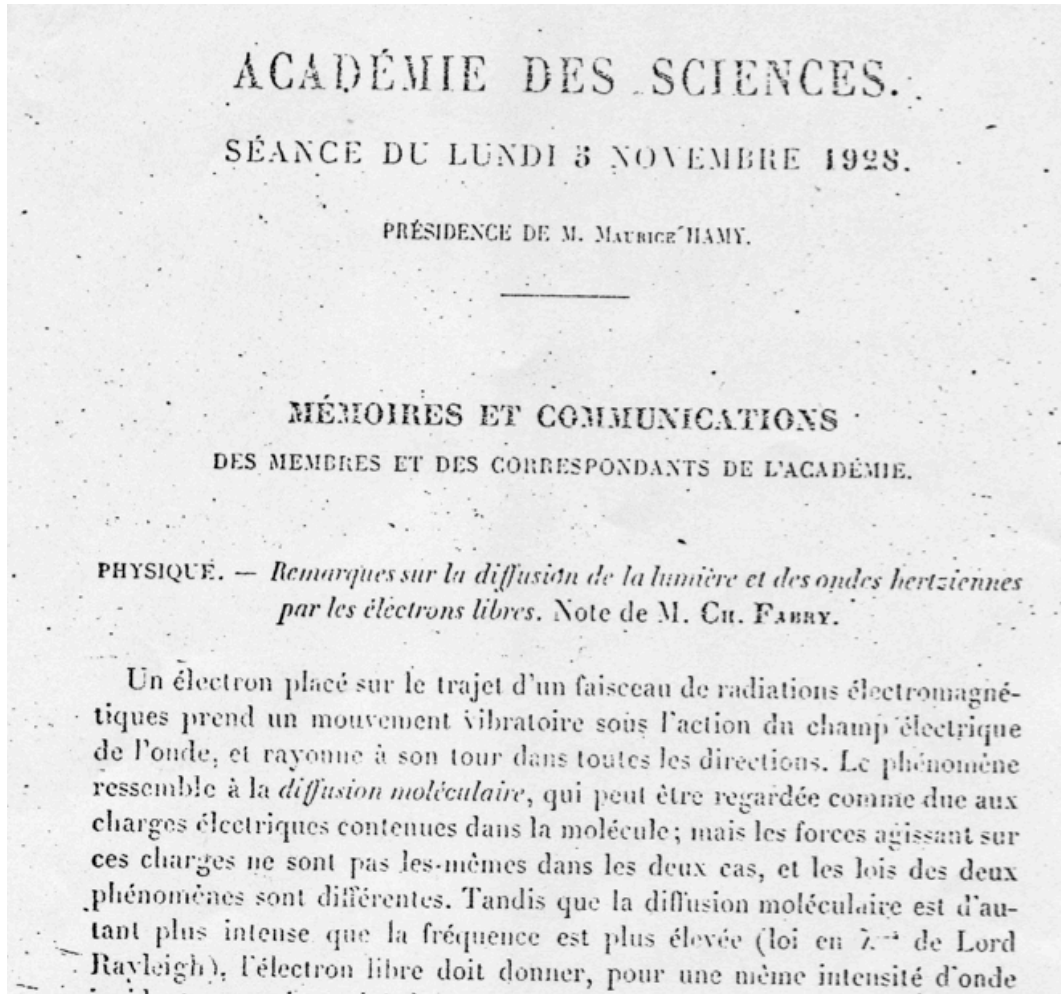
$$S/N \sim 5$$

Workable!

But you need a megawatt class transmitter and a huge antenna.

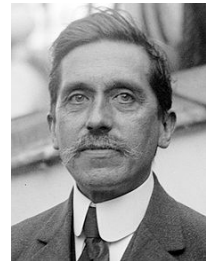
1950s: technology makes this possible (radio astronomy + construction = large antennas, military needs = high power transmitters)

Incoherent Scatter Concepts Are Older Than You Think



Remarques sur la diffusion de la lumière et des ondes hertziennes par les électrons libres

C. Fabry
1928



Charles Fabry
1867-1945

$$\sigma = \frac{8\pi}{3} \frac{e^4}{m^2 c^4} \omega^4$$

Electron scattering cross section
(fundamental)

Incoherent Scatter Concepts Are Older Than You Think

Without worrying about noise:

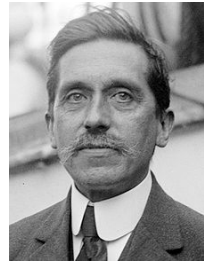
Rayleigh scattering $\propto \lambda^4$ [why is the sky blue?]

Incoherent scatter independent of wavelength
[but it's weak]

*Remarques sur la diffusion de la
lumière et des ondes hertziennes
par les électrons libres*

Incoherent Scatter concept!

C. Fabry
1928



Charles Fabry
1867-1945

For luminous radiations whose wavelength is very small, there is no phase relation between the elementary waves sent out by the different electrons of even a small volume and it is the intensities which add up. Thus, if a certain volume contains a total number of electrons n , then the power that it scatters is that transmitted by an area $S = n\sigma$. With the degrees of ionization that can actually exist, the scattering of light by electrons is always very slight. That is why it plays no appreciable role in the production of light in the diurnal sky *.

First Incoherent Scatter Radar

- W. E. Gordon of Cornell is credited with the idea for ISR.
- *“Gordon (1958) has recently pointed out that scattering of radio waves from an ionized gas in thermal equilibrium may be detected by a powerful radar.”* (Fejer, 1960)
- Gordon proposed the construction of the Arecibo Ionospheric Observatory for this very purpose (NOT for radio astronomy as the primary application)

~40 megawatt-acres



- 1000' Diameter Spherical Reflector
 - 62 dB Gain
- 430 MHz line feed 500' above dish
- Gregorian feed
- Steerable by moving feed.

Incoherent Scattering of Radio Waves by Free Electrons with Applications to Space Exploration by Radar*

W. E. GORDON†, MEMBER, IRE

INTRODUCTION

FREE electrons in an ionized medium scatter radio waves incoherently so weakly that the power scattered has previously not been seriously considered. The calculations that follow show that this incoherent scattering, while weak, is detectable with a powerful radar. The radar, with components each representing the best of the present state of the art, is capable of:

- 1) measuring electron density and electron temperature as a function of height and time at all levels in the earth's ionosphere and to heights of one or more earth's radii;
- 2) measuring auroral ionization;
- 3) detecting transient streams of charged particles coming from outer space; and
- 4) exploring the existence of a ring current.

* Original manuscript received by the IRE, June 11, 1958; revised manuscript received, August 25, 1958. The research reported in this paper was sponsored by Wright Air Dev. Ctr., Wright-Patterson Air Force Base, O., under Contract No. AF 33(616)-5547 with Cornell Univ.

† School of Elec. Eng., Cornell Univ., Ithaca, N. Y.



First Incoherent-Scatter Radar

- **K.L. Bowles [Cornell PhD 1955]**, Observations of vertical incidence scatter from the ionosphere at 41 Mc/sec. *Physical Review Letters* 1958:

“The possibility that incoherent scattering from electrons in the ionosphere, vibrating independently, might be observed by radar techniques has apparently been considered by many workers although seldom seriously because of the enormous sensitivity required...”

First Incoherent-Scatter Radar

...Gordon (W.E. Gordon from Cornell) recalled this possibility to the writer [spring 1958; D. T. Farley] while remarking that he hoped soon to have a radar sensitive enough to observe electron scatter in addition to various astronomical objects..."

Bowles executed the idea - hooked up a large transmitter to a dipole antenna array in Long Branch Ill., took a few measurements.

Gordon presenting on same day at October 21, 1958 Penn State URSI meeting:

"...And then I want to tell you about a telephone call that I just had."

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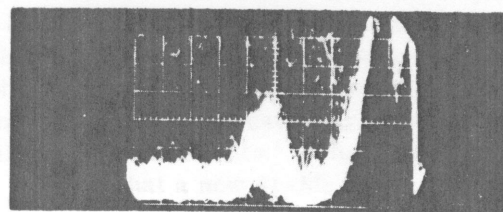
PHYSICAL REVIEW LETTERS

DECEMBER 15, 1958

Table I. Parameters of radar equipment used.

Operating frequency	40.92 Mc/sec
Peak pulse power	$(4 \text{ to } 6) \times 10^6$ watts
Pulse duration	$(50 \text{ to } 150) \times 10^{-6}$ sec
Average power	4×10^4 watts maximum
Receiver bandwidth	10, 15, or 30 kc/sec
Antenna cross section	116×140 meters (1024 half-wave elements in phase above ground)
Antenna polarization	north-south
Calculated antenna gain	~ 35 decibels/isotropic

~6 week setup time



Oscilloscope + camera + ~4 sec exposure
(10 dB integration)

FIG. 2. Pulse with 30 kc/sec receiver bandwidth
30 kc.

Incoherent Scattering Detectability

Bowles' results found approximately the expected amount of power scattered from the electrons (scattering is proportional to charge to mass ratio - electrons scatter the energy).

BUT: his detection with a 20 megawatt-acre system at 41 MHz (high cosmic noise background; should be marginal) implies a spectral width 100x narrower than expected – almost as if the much heavier (and slower) ions were controlling the scattering spectral width.


In fact, they do.

Calculating the fluctuation spectrum

We need to find a way to calculate electron plasma fluctuations at a given wavelength and frequency - use statistical mechanics:

Equipartition of kT of energy between potential, kinetic energy; normal mode = 2 traveling waves with equally divided energy

$$\langle KE \rangle_{wave} = \frac{1}{4} k_B T = \frac{1}{2} N_0 m \langle |v(\vec{k}, \omega)|^2 \rangle$$


$$\langle |v(\vec{k}, \omega)|^2 \rangle = \frac{1}{2} \frac{k_B T}{N_0 m}$$

Add the continuity equation:

$$\frac{\partial N}{\partial t} + \nabla \cdot (N \vec{v}) = 0$$

Calculating the fluctuation spectrum

$$\left\langle \left| \Delta N(\vec{k}, \omega) \right|^2 \right\rangle = \frac{1}{2} N_0 \frac{k_B T}{m} \left(\frac{k^2}{\omega^2} \right)$$

Insert plasma dispersion relation here

We need the full dispersion relation expression.

Dispersion relation: the concept

Dispersion relation = relationship between oscillation frequency and wave vector.

Each governing physical process in the wave medium - plasma in this case - has its own set of equations which define this relationship.

Simplest case: uniform phase velocity

$$\omega(k) = c k$$

Most propagation speeds depend nonlinearly on the wavelength and/or frequency.

NB: for a nonlinear dispersion relation, the pulse will typically spread in either spatial frequency or temporal frequency as a function of time (dispersion).

Calculating the fluctuation spectrum: Review

$$\left\langle \left| \Delta N(\vec{k}, \omega) \right|^2 \right\rangle = \frac{1}{2} N_0 \frac{k_B T}{m} \left(\frac{k^2}{\omega^2} \right)$$

Insert plasma dispersion relation here

We need the full dispersion relation expression.

This is not a plasma waves course so we won't derive it, but the two most important modes are:

1) *Ion-acoustic fluctuations* [sound waves in plasma]

$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = V_s$$

NB: ordinary acoustic waves:
adiabatic compression /
decompression of fluid particles.

Ion-acoustic fluctuations:
restoring force = electromagnetic

Calculating the fluctuation spectrum

$$\left\langle \left| \Delta N(\vec{k}, \omega) \right|^2 \right\rangle = \frac{1}{2} N_0 \frac{k_B T}{m} \left(\frac{k^2}{\omega^2} \right)$$

Insert plasma dispersion relation here

We need the full dispersion relation expression.

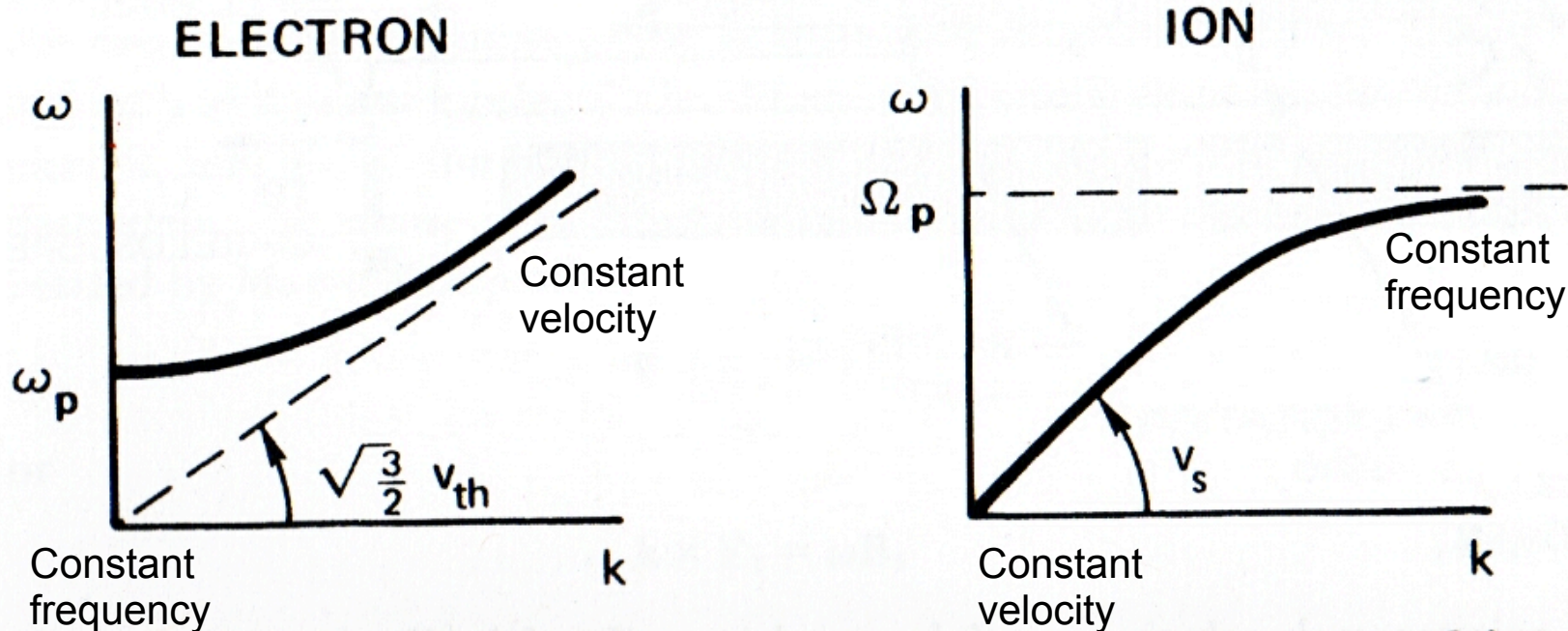
This is not a plasma waves course so we won't derive it, but the two most important modes are:

2) *Langmuir oscillations* (Plasma oscillations):

$$\omega^2 = \omega_p^2 + \frac{3}{2} k^2 v_{th}^2 \quad v_{th}^2 = 2k_B T_e / m_e$$

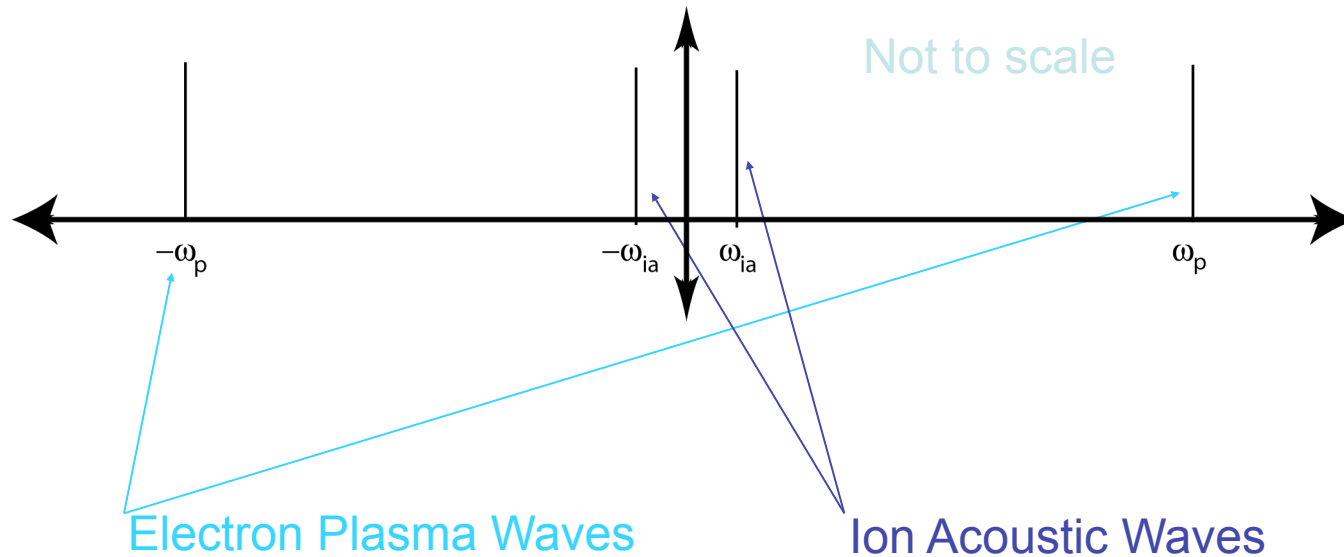
Akin to Brunt-Väisälä oscillations in fluid (parcel in presence of density gradient) - here, electrostatic field is restoring force, and electron pressure gradient transmits information

Electron and ion waves: Dispersion relations



Comparison of the dispersion curves for electron plasma waves and ion acoustic waves.

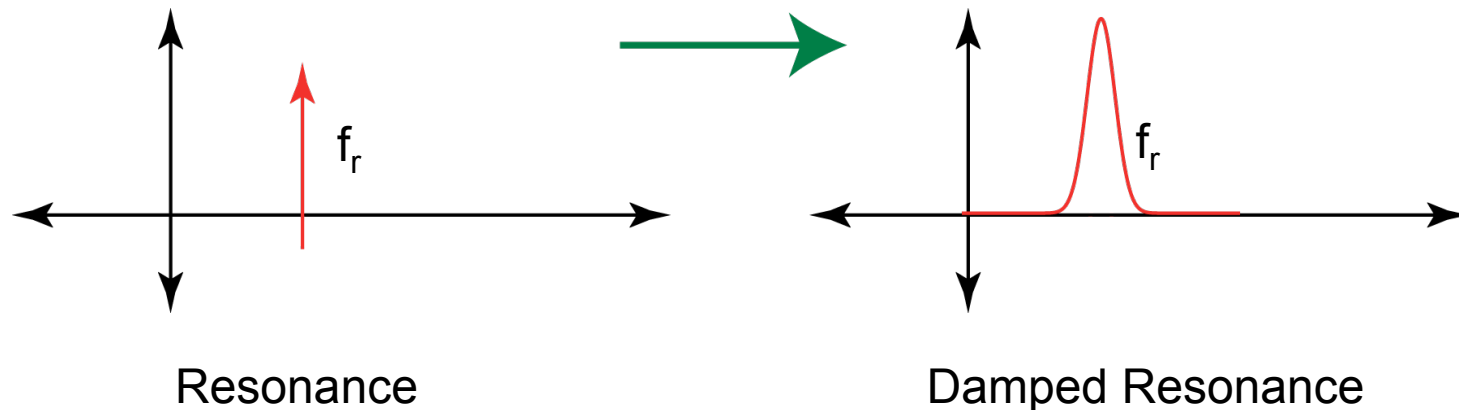
Wave Spectrum



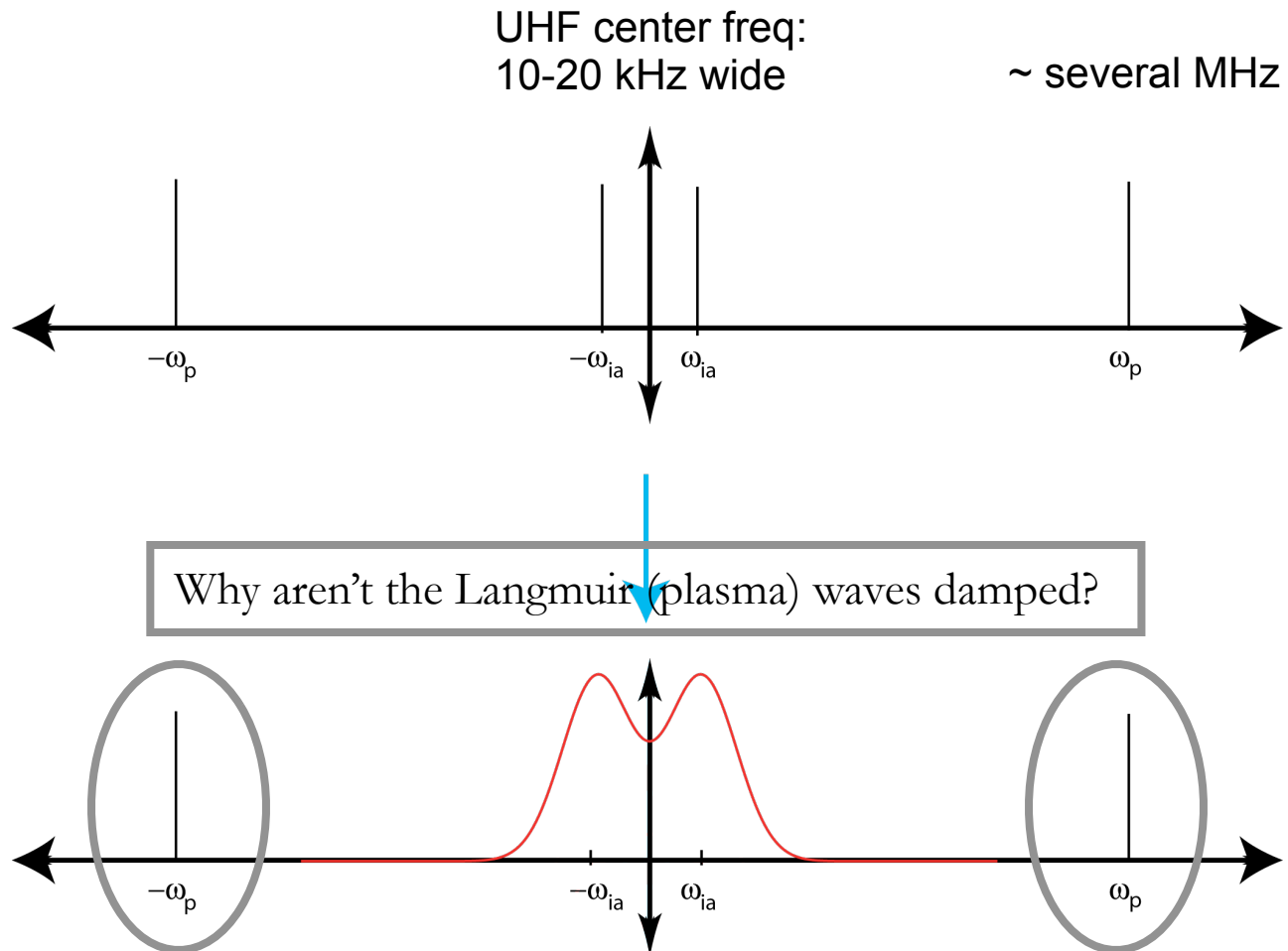
Plasma parameters fluctuate with the waves (density, velocity, etc)

Damped resonance

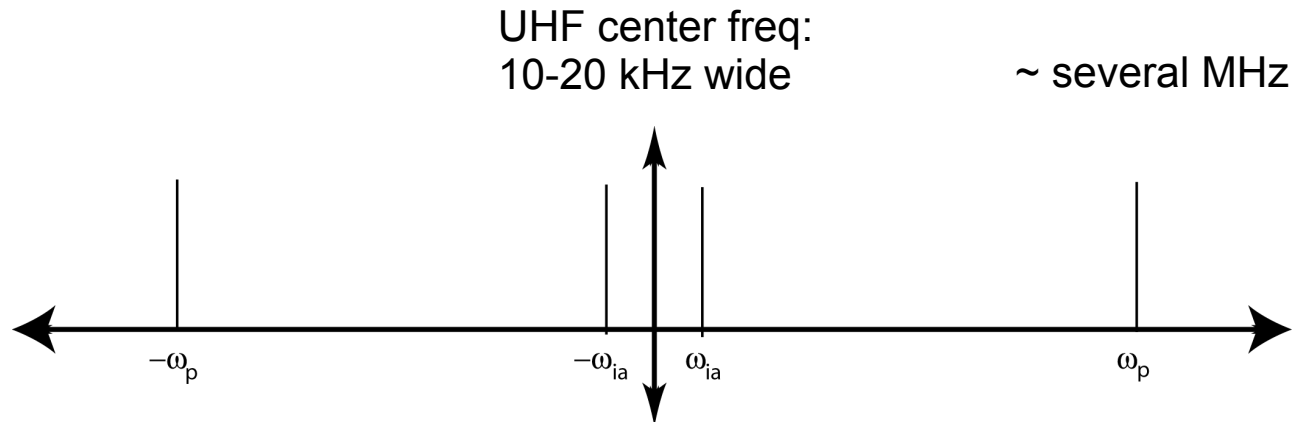
- Waves in a plasma are resonances.
- Damped resonances are not sharp
 - Example – Q of a resonant circuit.
- IS: Thermal ions have motions close to ion-acoustic speed (Landau damping – “surfing”; locked to I-A waves)



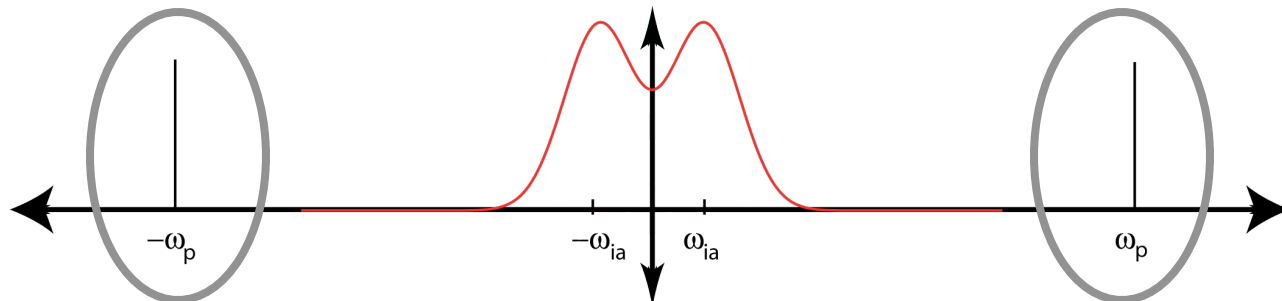
Wave Spectrum (ISR Spectrum)



Wave Spectrum (ISR Spectrum)



Why aren't the Langmuir (plasma) waves damped?
Electron thermal velocity ~ 125 km/s but plasma wave frequency \sim several MHz –
Not much interaction and not much damping.



Nyquist-Johnson thermal noise (1928): Always present

JULY, 1928

PHYSICAL REVIEW

VOLUME 32

THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

By H. NYQUIST

ABSTRACT

The electromotive force due to thermal agitation in conductors is calculated by means of principles in thermodynamics and statistical mechanics. The results obtained agree with results obtained experimentally.



H. Nyquist 1889-1976
(born Nilsby, Sweden)

“Bert” Johnson 1887-1970
(born Gothenburg, Sweden)



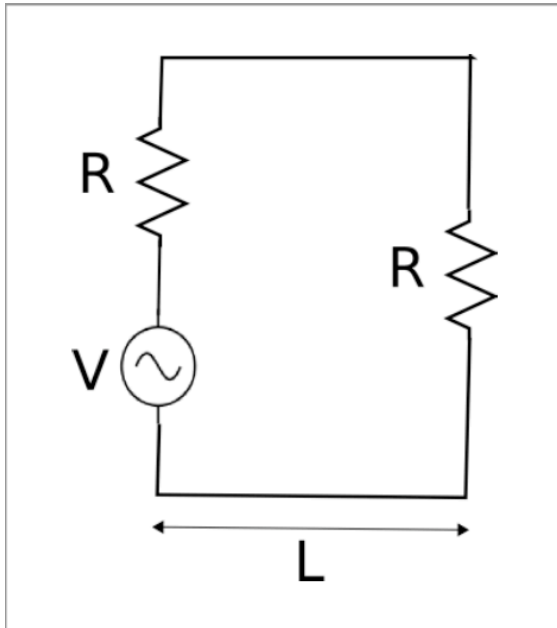
Nyquist-Johnson: The Motivation





Nyquist-Johnson thermal noise (1928)

Circuit loop has a temperature T
(with thermal vibrations)



(Each resistor)

$$P_{absorbed} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{h|\omega|}{e^{\frac{h|\omega|}{k_B T}} - 1}$$

$$I = \frac{V}{2R}$$

$$P_{emitted} = \frac{\langle V^2 \rangle}{4R} \leftarrow S(\omega)$$

Voltage spectral
density in the circuit
loop (= per frequency):

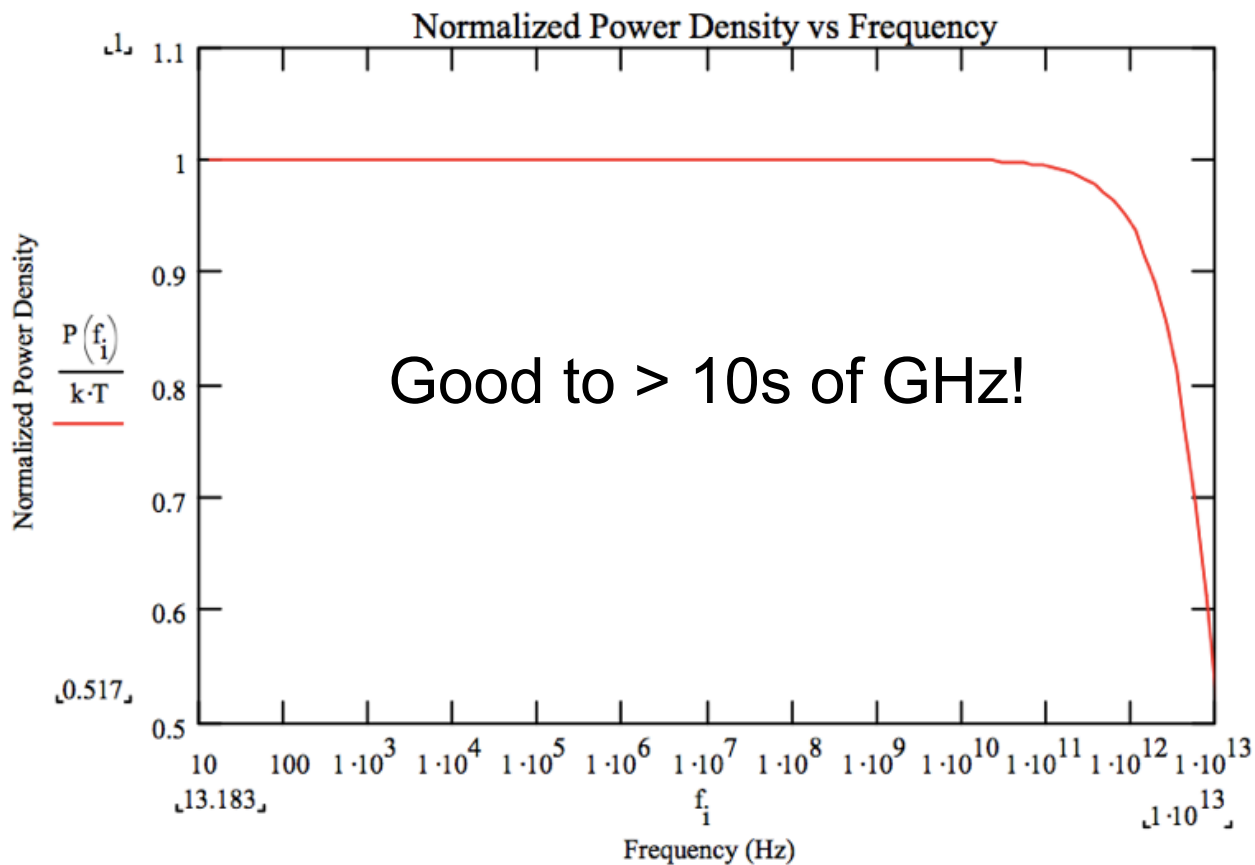
$$S(\omega) = 4R \frac{h|\omega|}{e^{\frac{h|\omega|}{k_B T}} - 1}$$

$$h\omega \ll k_B T : S(\omega) = 4Rk_B T$$

Over a range of
frequencies:

$$P(\Delta f) \propto k_B T \Delta f$$

Nyquist-Johnson thermal noise (1928)



(Clay Turner, Wireless Solutions, 2007)

Nyquist generalization: Fluctuation-dissipation theorem

Powerful tool in statistical physics.
Fundamental assumption:

If our system is in thermal equilibrium (*),
The response we see to a small applied force is the same as its
response to a spontaneous fluctuation (i.e. statistical fluctuations in
equilibrium).

Nyquist was using this principle for the derivation we have just seen.

It can be proven and generalized.

(*) There are also modifications for it to work in cases where thermal
equilibrium does not exist - e.g. multiple species or populations, each
with its own thermal distribution function.

Nyquist generalization: Fluctuation-dissipation theorem

Callen and Welton [1951]:

Any linear dissipative system whose components are in thermal equilibrium will exhibit thermally driven fluctuations having power spectra which can be derived by applying Nyquist noise principles to an equivalent circuit model system.

We therefore need to derive such a model for thermal plasma.

Ampere's law:
$$-j\vec{k} \times \vec{H} = \vec{J} + j\omega\epsilon_0\vec{E}$$

becomes
(in \vec{k} dir)

$$0 = (\sigma_i + \sigma_e)E + \frac{\omega}{k}e(n_{th,i} - n_{th,e}) + j\omega\epsilon_0E$$

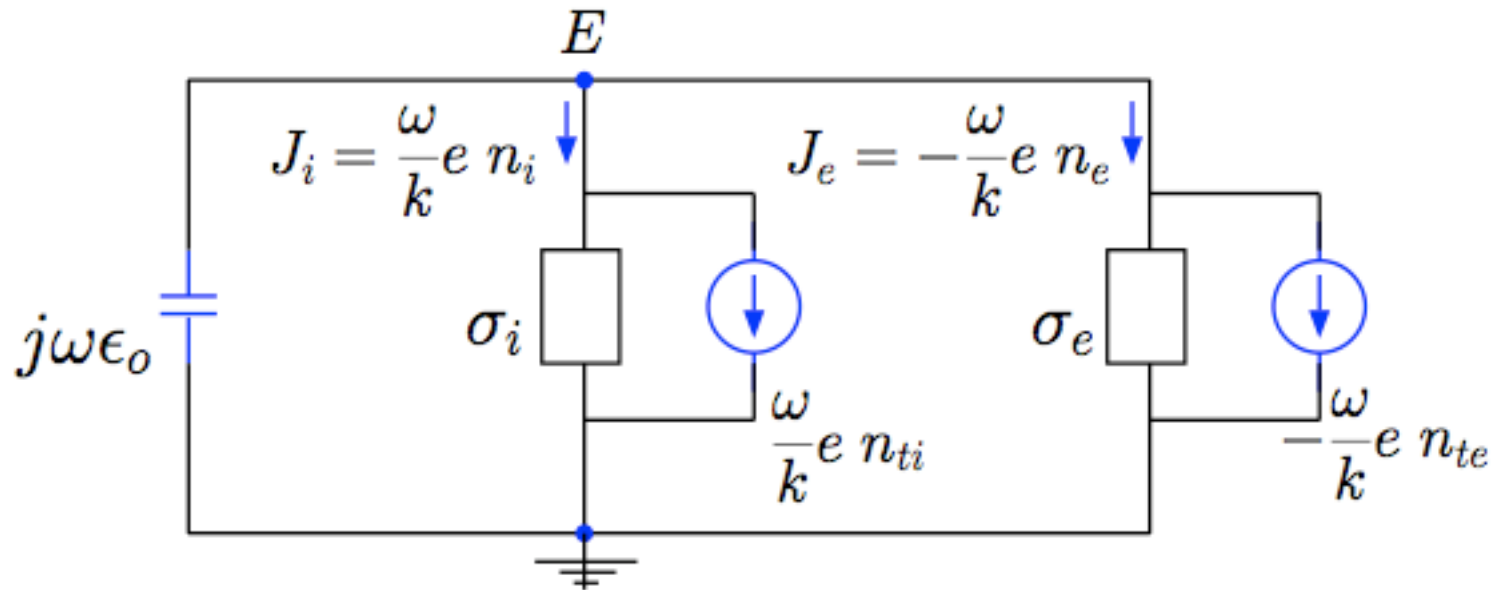
Conduction currents
(particle motion deviations)

Thermal random currents
(and Poisson's law for
overall plasma neutrality)

Displacement current

Fluctuation-dissipation: Plasma Circuit Model

$$0 = (\sigma_i + \sigma_e)E + \frac{\omega}{k}e(n_{th,i} - n_{th,e}) + j\omega\epsilon_0 E$$



Solve for E and use in current continuity to get density fluctuations

$$-\frac{\omega}{k}en_e = E\sigma_e - \frac{\omega}{k}en_{th,e}$$

Electron density fluctuation spectrum

Solution yields

$$\left\langle \left| n_e(\vec{k}, \omega) \right|^2 \right\rangle = \frac{|j\omega\epsilon_0 + \sigma_i|^2 \left\langle \left| n_{th,e}(\vec{k}, \omega) \right|^2 \right\rangle}{|j\omega\epsilon_0 + \sigma_e + \sigma_i|^2} \quad \text{Electron line}$$
$$+ \frac{|\sigma_e|^2 \left\langle \left| n_{th,i}(\vec{k}, \omega) \right|^2 \right\rangle}{|j\omega\epsilon_0 + \sigma_e + \sigma_i|^2} \quad \text{Ion line}$$

Beyond this point, use plasma kinetic theory to derive ion and electron conductivities in a thermal plasma.

cf. Dougherty and Farley (1960) and “Farley series”

Compare to other approaches: Fejer (1960), Salpeter (1960), Hagfors (1961)

Electron density fluctuation spectrum

Use electron force/response concept and solve for electron and ion admittances y_e, y_i (analogous to resistive dissipation). Arrive at spectral expression

$$\sigma_0(\omega_o + \omega)d\omega = N_0 r_e^2 \operatorname{Re} \left\{ \frac{y_e(y_i + jk^2 \lambda_{de}^2)}{y_e + y_i + jk^2 \lambda_{de}^2} \frac{d\omega}{\pi\omega} \right\}$$

- Short wavelength limit ($k^2 \lambda_{de}^2 \gg 1$): pure e^- scatter
- Long wavelength limit: RHS $\rightarrow y_e y_i / (y_e + y_i)$: damped ion-acoustic resonances
- Near plasma frequency: $y_e + y_i + jk^2 \lambda_{de}^2 \rightarrow 0$: plasma lines

Incoherent Scatter Spectral Dependence

Spectral response can be evaluated using these frameworks for:

- Thermal inequality $T_e \neq T_i$: decreases Landau damping
- Ion-neutral collisions ν_{in} : narrows spectrum
- Background magnetic field B_0 : makes electrons heavier

$$m_e \rightarrow m_e^* = \frac{m_e}{\cos^2 \alpha}$$

Also, ion gyro-resonance (mass-dependent).

Incoherent Scatter Spectral Dependence

- Ion mixtures: $\frac{T_e}{T_i} y_i \rightarrow \sum_j \frac{T_e}{T_j} \frac{N_j}{N_0} y_j(m_j, T_j)$
- Unequal ion temperatures
- Particle drifts: $\omega \rightarrow \omega - \vec{k} \cdot \vec{v}_{de}$
- Plasma line measurements ($[e^-]$, T_e , $v_{||}$)
- Photoelectron heating, non-Maxwellian plasmas
- Faraday rotation effects (equator, low TX freq)

Things can get hairy. For example, magnetic field evaluation requires Gordeyev integral:

$$\int e^{j(\theta - j\phi)t - \frac{\sin^2 \alpha}{\phi^2} t} \sin^2\left(\frac{\phi t}{2}\right) - \frac{t^2}{4} \cos^2 \alpha dt$$

IS Spectral Shape Demonstration

(See IS Spectrum Java applet on “ISR Demonstration” page)

Measurement Statistics

$E_s(t)$ and $\therefore V_s(t)$ are Gaussian random variables
(Central Limit Theorem):

$$\begin{aligned}V_s(t) &= V_1 &&= x_1 + jx_2 \\V_s(t + \tau) &= V_2 &&= x_3 + jx_4\end{aligned}$$

We desire ensemble averages of 2nd moments
(correlations):

$$\langle V_1 V_2^* \rangle = \langle (x_1 + jx_2)(x_3 + jx_4)^* \rangle = S\rho(\tau)$$

where S is signal power, and IS theory gives medium
correlation

$$\rho(\tau) = \rho_R(\tau) + j\rho_I(\tau)$$

Measurement Statistics

In general, we define an estimator to approximate true ensemble average - e.g.

$$\hat{S} = \frac{1}{K} \sum_{i=1}^K V_i V_i^*$$

might be power estimator for true $S = \langle V_1 V_1^* \rangle$.
Each estimator will have an associated bias and variance, e.g.

$$\text{bias} = \langle \hat{S} \rangle$$

$$\text{variance}(\hat{S} - S) = \langle (\hat{S} - S)^2 \rangle$$

Power Estimation

For total scattered power, use

$$\hat{S} = \frac{1}{K} \sum_{i=1}^K V_i V_i^*$$

$$\text{Bias : } \hat{S} = S$$

$$\text{Variance : } \langle (\hat{S} - S)^2 \rangle = \frac{S^2}{K} = \delta_S^2$$

$$\text{RMS frac error : } \frac{\delta_S}{S} = \frac{1}{\sqrt{K}}$$

10,000 samples needed for 1% accuracy.

Power Estimation: Noise Effects

Add noise (Gaussian RV with different 2nd moment).
Use estimator

$$\hat{S} = S + \hat{N} - \hat{N}$$

$$\text{Bias : } \hat{S} = S$$

$$\text{Variance : } \delta_S^2 \sim \frac{(S + N)^2}{K_{S+N}}$$

$$\text{RMS frac error : } \frac{\delta_S}{S} \sim \frac{S + N}{S} \frac{1}{\sqrt{K_{S+N}}}$$

ACF Estimation

We want $\langle V(t)V^*(t + \tau) \rangle = \langle V_1 V_2^* \rangle = S\rho(\tau)$. A popular estimator is:

$$\hat{\rho} = \frac{\frac{1}{K} \sum_{i=1}^K V_{1i} V_{2i}^*}{\left[\frac{1}{K^2} \sum_{i=1}^K |V_{1i}|^2 \sum_{i=1}^K |V_{2i}|^2 \right]^{\frac{1}{2}}} = \frac{A}{B}$$

After linearizing and lots of details:

$$\text{Bias : } \hat{\rho} = \rho \left(1 - \frac{1}{4K} (1 - |\rho|^2) \right)$$

$$\text{Variance : } \delta_{\rho}^2 = \frac{1}{K} \left[1 - \frac{3}{2} |\rho|^2 + \frac{1}{2} |\rho|^4 \right]$$

ACF Estimation: Noise Effects

Effect of adding noise is to change the estimator:

$$\hat{\rho} = \frac{A_{S+N} - A_N}{B_{S+N} - B_N}$$

Details show that

$$\delta_{\hat{\rho}}^2 \sim \frac{1}{K} \left(\frac{S+N}{S} \right)^2 \left[1 + \frac{1}{2} |\rho_S|^2 \right]$$

Consequences:

- When SNR low, variance large
- Larger S is wasted statistically