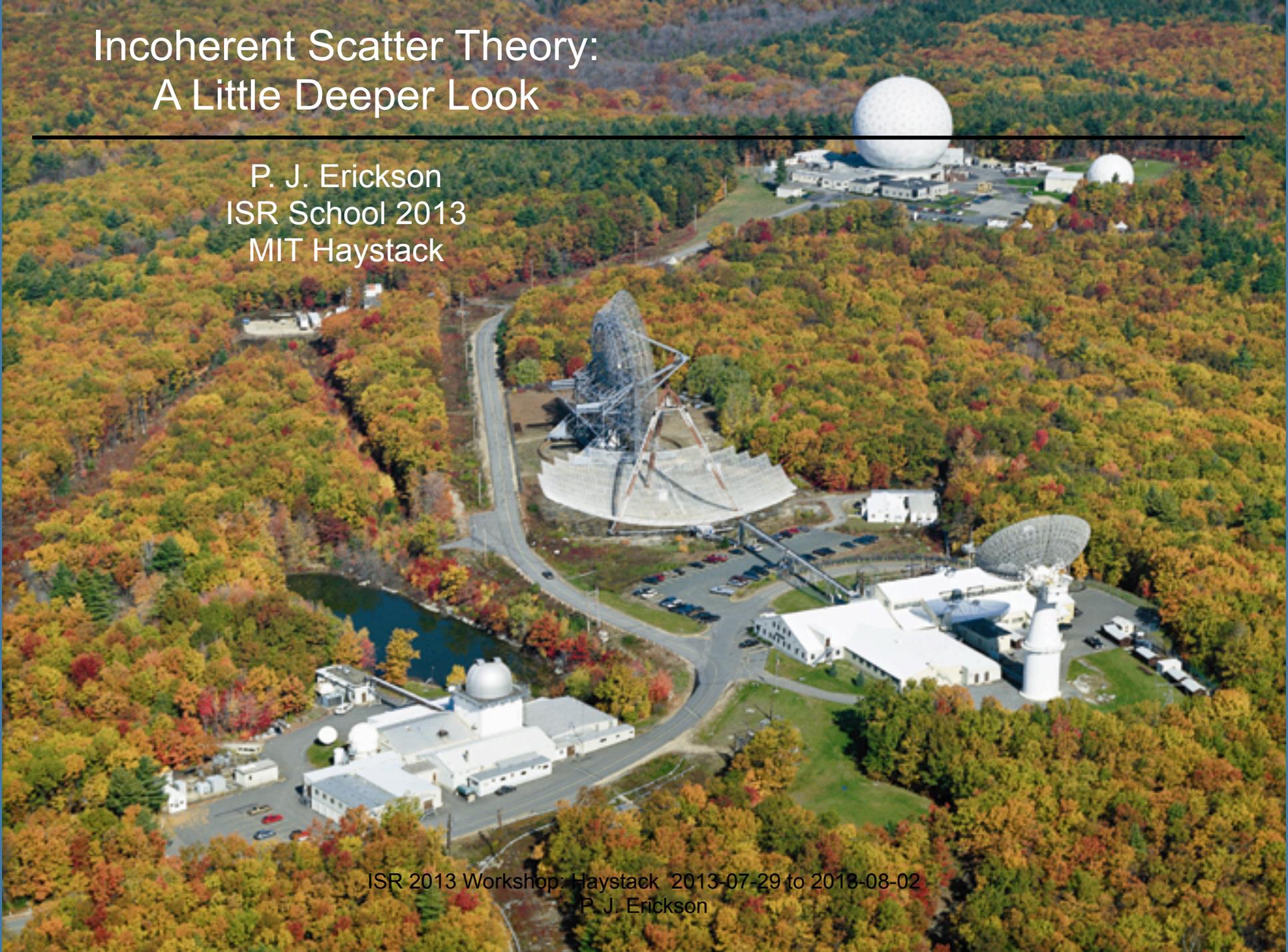


Incoherent Scatter Theory: A Little Deeper Look

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ISR School 2013
MIT Haystack

ISR 2013 Workshop: Haystack 2013-07-29 to 2013-08-02
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Maxwell's Equations



J. C. Maxwell
1831 - 1879

Governs propagation of electromagnetic waves ("action at a distance"), relation between electric and magnetic field and motions of charges

Foundation of classical electromagnetic theory

Gauss' Law (electric field around charges)

$$\nabla \cdot \mathbf{D} = \rho_f$$

in free space:
 $\mathbf{H} = \mathbf{B}$ $\mathbf{D} = \mathbf{E}$

Gauss' Law for magnetism (no magnetic monopoles)

$$\nabla \cdot \mathbf{B} = 0$$

Faraday's Law (electric field around a changing magnetic field)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Ampere's Law (magnetic field circulation around electric charges)

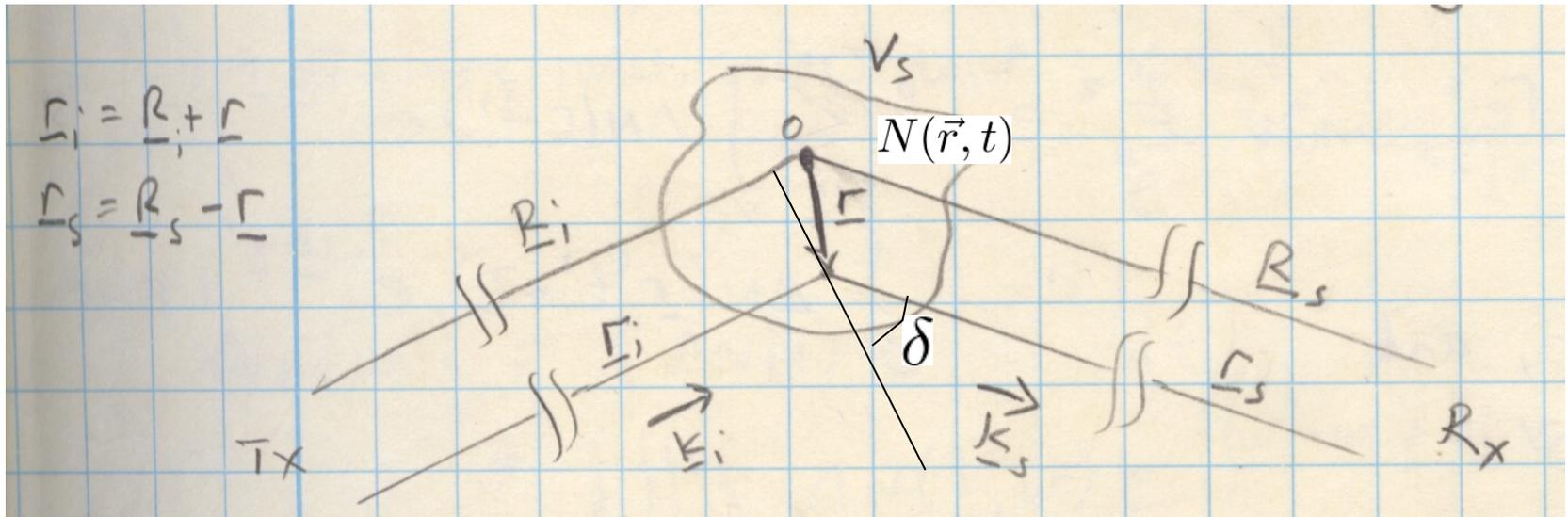
$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

↑
Maxwell's correction
(displacement current)

Single Electron Scattering Model

Incident EM wave accelerates each charged particle it encounters. These then re-radiate an EM wave.

For a single electron located at $r = 0$, we need the scattered field at a distance r_s .



Single Electron Scattering Model

Incident EM wave accelerates each charged particle it encounters.
These then re-radiate an EM wave.

For a single electron located at $r = 0$, the scattered field at a distance r_s :

$$\text{scattered field} \quad \left| \vec{E}_s(\vec{r}_s, t) \right| = \frac{e^2 \mu_0 \sin \delta}{4\pi R m_e} \left| \vec{E}_i(0, t') \right| \quad \text{Incident field}$$

$$= \frac{r_e}{R} \sin \delta \left| \vec{E}_i(0, t') \right|$$

$$r_e = \frac{e^2 \mu_0}{4\pi m_e} \quad \text{Classical electron radius}$$

$$t' = t - \frac{R}{c} \quad \text{Delayed time}$$

$$\sin \delta \quad \text{Scattering angle}$$

Scattering Model

Assume a volume filled with electron scatterers whose density is represented in space and time by

$$N(\vec{r}, t)$$

Illuminating this volume with an incident field from a transmitter location means that each electron contributes to the resulting scattered field, using *Born approximation* (each scatter is weak and does not affect others).

With geometrical considerations, scattered field at receiver location is now:

$$E_s(t) = r_e \sin \delta E_0 e^{j\omega_0 t} \int_{V_s} \frac{1}{r_s} N(\vec{r}, t') e^{-j(\vec{k}_i - \vec{k}_s) \cdot \vec{r}} d^3 \vec{r}$$

$$t' = t - \frac{r_i}{c} \quad \text{Delayed time}$$

Scattering Model

Assume densities have random spatial and temporal fluctuations about a background:

$$N(\vec{r}, t) \rightarrow N_0 + \Delta N(\vec{r}, t)$$

Further, assume backscatter (i.e. monostatic radar):

$$\vec{k} = 2\vec{k}_i \qquad r_i \equiv r_s = R$$

Then, scattered field reduces to:

$$E_s(t) \rightarrow \frac{r_e}{R} \sin \delta E_0 e^{j\omega_0 t} \underbrace{\int_{V_s} \Delta N(\vec{r}, t') e^{-j\vec{k} \cdot \vec{r}} d^3\vec{r}}_{\equiv \Delta N(\vec{k}, t')}$$

Scattering Model

Plasmas (ionosphere) are thermal gases and $\Delta N(\vec{r}, t)$ is a Gaussian random variable, so the Central Limit Theorem applies:

statistical average \longrightarrow $\langle E_s(t) \rangle = \langle \Delta N(\vec{r}, t) \rangle = 0$

It's much more useful to look at second order products – in other words, examine temporal correlations in the scattered field:

$$\langle E_s(t) E_s^*(t + \tau) \rangle \propto e^{-j\omega_0\tau} \langle \Delta N(\vec{k}, t) \Delta N^*(\vec{k}, t + \tau) \rangle$$

Useful things to measure can now be defined.

Scattering: Measurable Quantities

Defining $C_s = \frac{r_e^2 E_0^2 \sin^2 \delta}{R^2} V_s$, then

Total scattered power

$$\langle |E_s(t)|^2 \rangle = C_s \langle |\Delta N(\vec{k})|^2 \rangle$$

and Autocorrelation function (ACF):

$$\langle E_s(t) E_s^*(t + \tau) \rangle = C_s e^{-j\omega_0 \tau} \langle \Delta N(\vec{k}, t) \Delta N^*(\vec{k}, t + \tau) \rangle$$

or Power Spectrum:

$$\langle |E_s(\omega_0 + \omega)|^2 \rangle \propto C_s \langle |\Delta N(\vec{k}, \omega)|^2 \rangle$$

Incoherent Scattering Model: Summary

Radar filters in k space:

$$\Delta N(\vec{r}, t) \rightarrow \Delta N(\vec{k}_r, t)$$

$$\Delta N(\vec{k}_r, t) \propto E_s(t)$$

Form ACF of $E_s(t)$ for each range, average, transform:

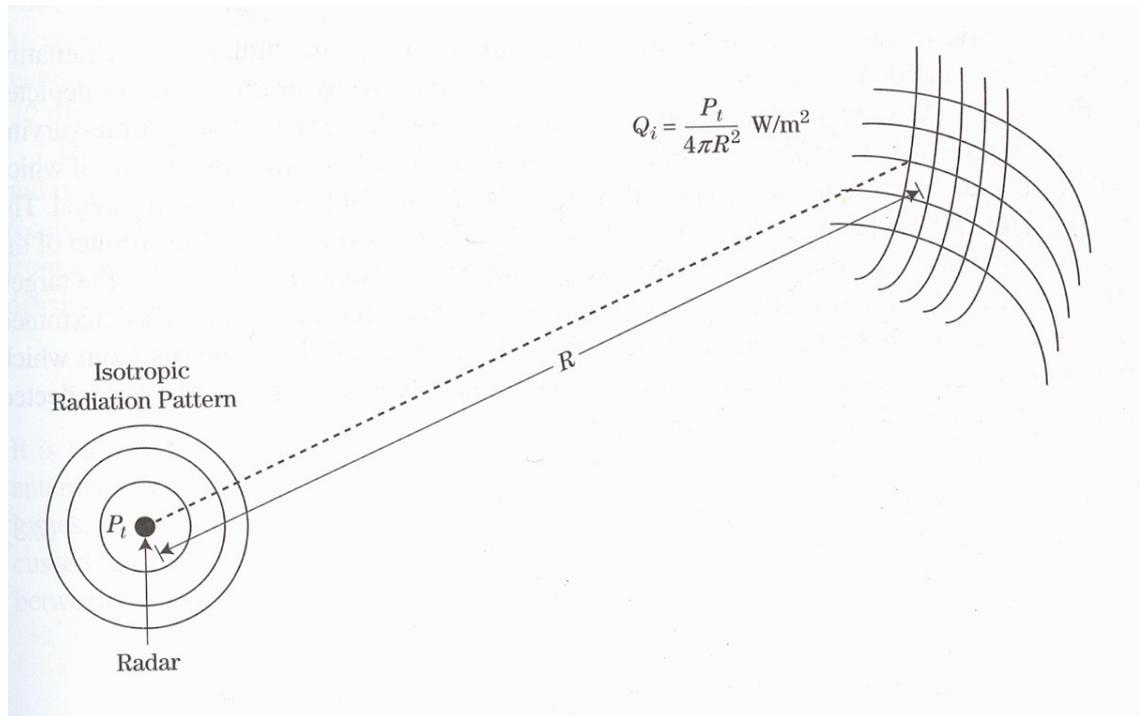
$$\langle E_s(t) E_s^*(t + \tau) \rangle \rightarrow \left\langle \left| \Delta N(\vec{k}, \omega) \right|^2 \right\rangle$$

Interpret latter in terms of the medium parameters.

The Radar Equation: Monostatic Version

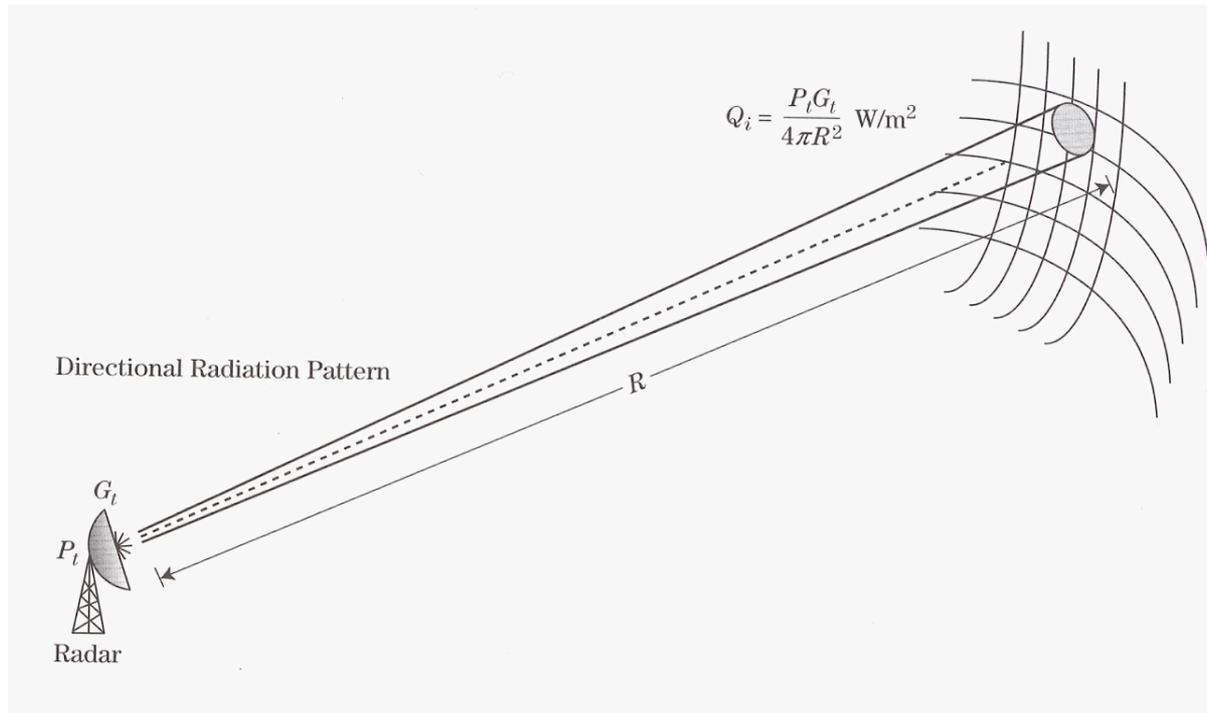
Power density at range R (isotropic):

$$\frac{P_t}{4\pi R^2}$$



The Radar Equation: Monostatic Version

Power density at range R (directional): $\frac{P_t G}{4\pi R^2}$

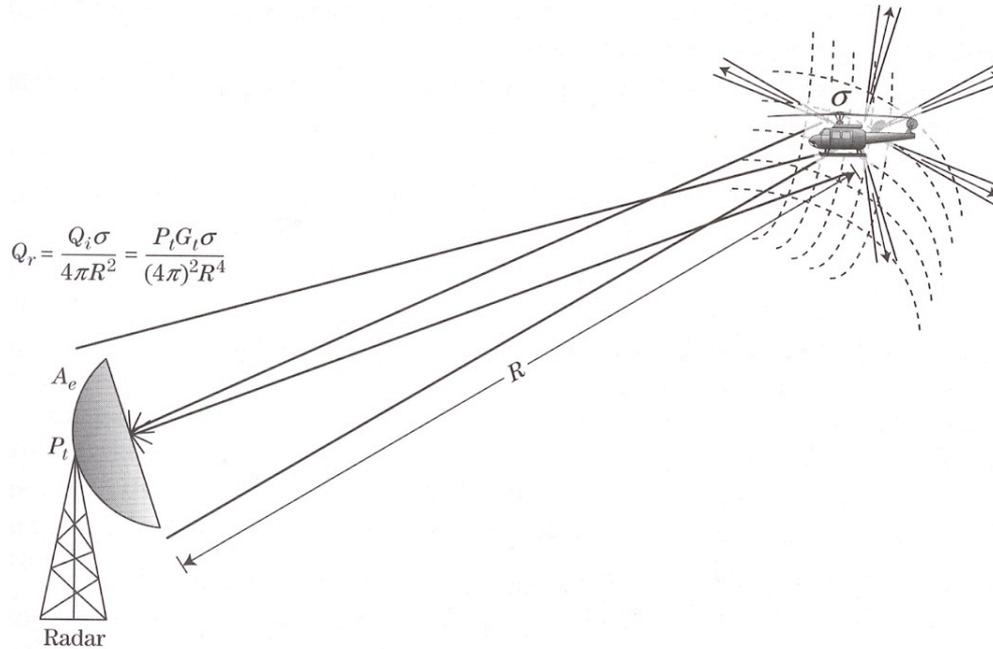


The Radar Equation: Monostatic Version

Radar cross section
(m²)

Reradiated power density at Rx:

$$\frac{P_t G}{4\pi R^2} \cdot \frac{\sigma}{4\pi R^2}$$



The Radar Equation: Monostatic Version

Total received power:
$$P_r = \frac{P_t G}{4\pi R^2} \cdot \frac{\sigma}{4\pi R^2} \cdot A_e = \frac{P_t G A_e \sigma}{(4\pi)^2 R^4}$$

Use gain/area relation -

The Radar Equation:

$$P_r = P_t \frac{\rho_a^2 A^2}{4\pi \lambda^2 R^4} \sigma$$

Maximum range form:

$$R_{max} = \left[\frac{P_t}{S_{min}} \frac{\rho_a^2 A^2}{4\pi \lambda^2} \sigma \right]^{\frac{1}{4}}$$

Hard vs Soft Radar Targets

Generalize radar equation for one or more scatterers, distributed over a volume:

$$P_r = \int P_t \frac{\rho_a^2 A^2}{4\pi\lambda^2 R^4} \sigma(\vec{x}) dV_s$$

First case: single scatterer (“hard target”) at single point in space:

$$\int \sigma(\vec{x}) dV_s = \sigma_{target} \equiv \sigma$$

Hard target
radar
equation:

$$P_r = P_t \frac{\rho_a^2 A^2}{4\pi\lambda^2 R^4} \sigma$$

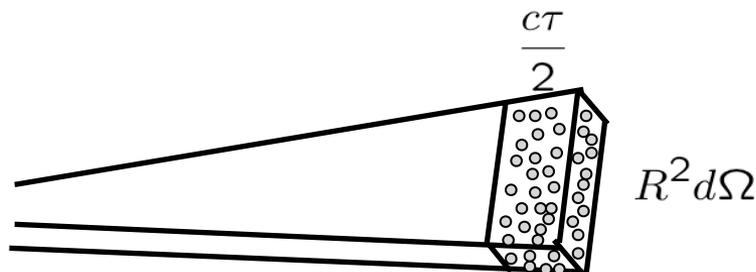


Sputnik 1 (1957-10-04)

Distributed Targets

$$\int \sigma(\vec{x}) dV_s = \int_0^{2\pi} \int_0^\pi \sigma(\vec{x}) \frac{c\tau}{2} R^2 d\Omega$$

$$\int \sigma(\vec{x}) dV_s = \frac{c\tau}{2} \int_0^{2\pi} \int_0^\pi \sigma(\vec{x}) R^2 \sin \theta d\theta d\phi$$



Assume volume is filled
with identical, isotropic
scatters

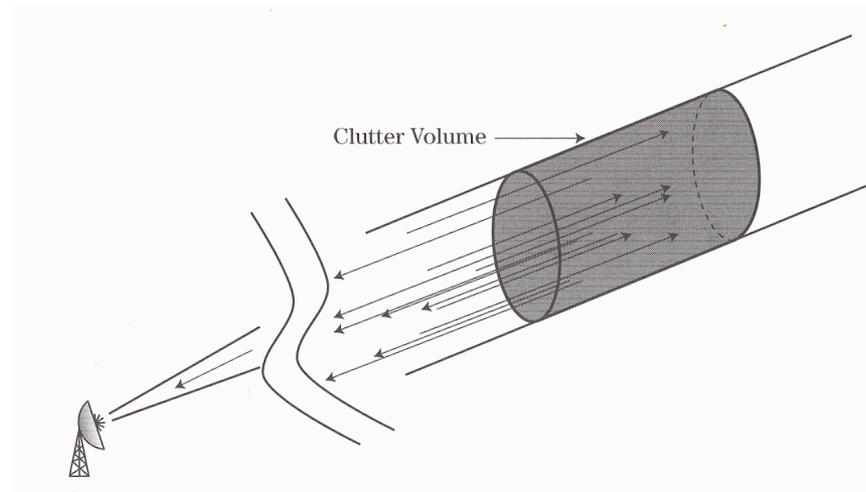
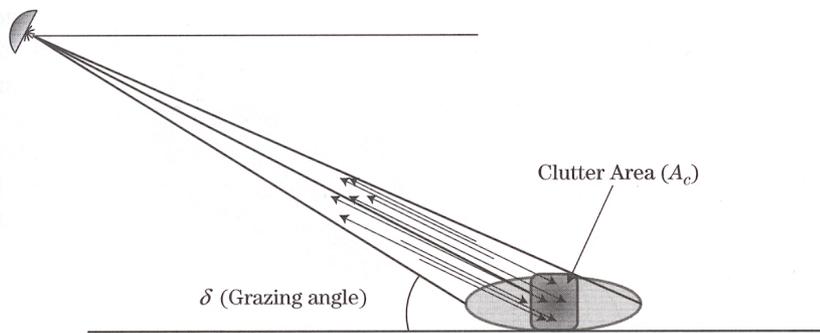
$$\int \sigma(\vec{x}) dV_s = \frac{c\tau}{2} R^2 \sigma$$

Distributed Scatterers

$$P_r = P_t \frac{\rho_a^2 A^2}{4\pi\lambda^2 R^4} \sigma \frac{c\tau}{2} R^2$$

The “soft target” Radar Equation

$$P_r = P_t \frac{c\rho_a^2 A^2 \tau}{8\pi\lambda^2 R^2} \sigma$$



Remote sensing a plasma: The experimental (radar) view

Suppose we transmit a wave towards a plasma and measure the scattered wave:

$$P_{rec} = (P_{inc}) A_{scat} \left(\frac{A_{rec}}{4\pi R^2} \right)$$

$$A_{scat} = \sigma_{radar} V_s \quad (\text{ionosphere is a beam filling target})$$

$$\sigma_{radar} = 4\pi \sigma_{total} \quad (\text{Solid angle})$$

$$\left(\frac{P_{rec}}{P_{inc}} \right) \left(\frac{4\pi R^2}{A_{rec}} \right) \left(\frac{1}{V_s} \right) = 4\pi r_e^2 \sin^2 \delta \langle |\Delta N(k)|^2 \rangle$$

Measurable experimentally

Detectability of scatter from ionospheric plasma

Assume a beam filling plasma at F region altitudes (300 km) with very high electron density (1E12 electrons per m³):

Classical electron scattering cross-section $\sigma_e = 10^{-28} m^2 / e^-$

Assume a pulse length of 10 km.

Assume a cross-beam width of 1 km (~ Arecibo).

$$\sigma_{tot} \sim 10^{-6} m^2$$

NB: Born approximation is very valid, since total amount of scattered power in the volume ~ 1E-12

Detectability of scatter from ionospheric plasma

For fraction of scattered power actually received, assume isotropic scatter and a BIG 100 m class antenna:

$$f_{rec} = \frac{A_{rec}}{4\pi R^2} \sim \frac{10^4 m}{4(300 \times 10^3 m)^2}$$

About -80 dB (1E-8): not much. So:

$$\frac{P_{rec}}{P_{tx}} \sim 10^{-20}$$

So a radar with 1 MW transmitted signal receives 10 femtowatts of incoherently scattered power from free electrons in the ionosphere.

REALLY not very much.

Detectability of scatter from ionospheric plasma

What matters, though, is the signal to noise ratio:

$$P_{noise} = (k_B T_{eff}) (BW) \quad (\text{derived later})$$

Typical effective noise temperatures ~ 100 to 200 K at UHF frequencies (430 MHz, say).

Assume the bandwidth is set by thermal electron motions in a Boltzmann sense:

$$3k_B T_e \sim m_e v_{e,th}^2$$

$$v_{e,th} \sim \sqrt{\frac{3k_B T_e}{m_e}} \sim 2 \times 10^5 \text{ m/s}$$

$$BW \sim (v_{e,th}) (2)(2) \left(\frac{f_{tx}}{c}\right) \sim 10^6 \text{ Hz}$$

(2s are for up/down, backscatter)

Detectability of scatter from ionospheric plasma

Finally,

$$P_{noise} \sim 2 \times 10^{-15} W$$

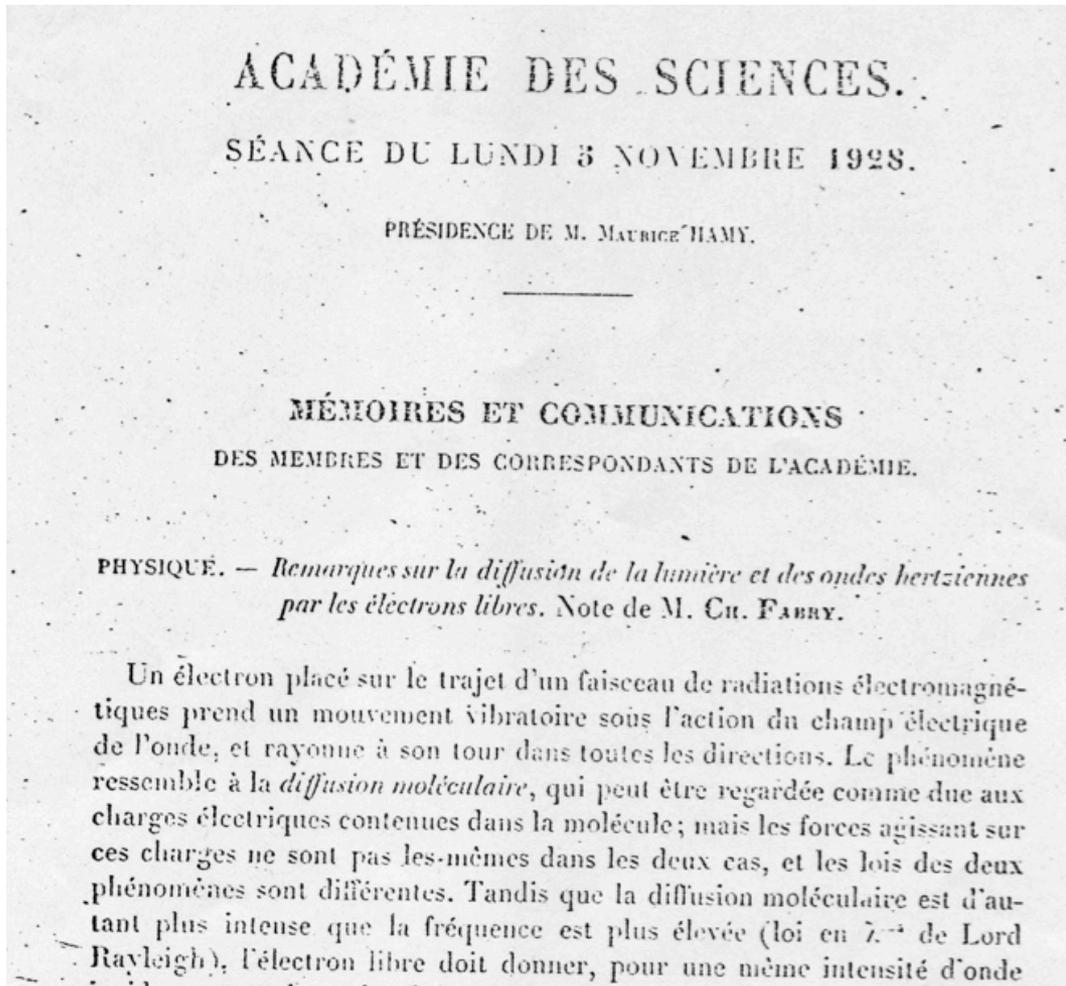
$$S/N \sim 5$$

Workable!

But you need a megawatt class transmitter and a huge antenna.

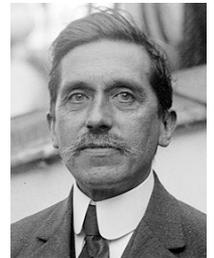
1950s: technology makes this possible (radio astronomy + construction = large antennas, military needs = high power transmitters)

Incoherent Scatter Concepts Are Older Than You Think



Remarques sur la diffusion de la lumière et des ondes hertziennes par les électrons libres

C. Fabry
1928



Charles Fabry
1867-1945

$$\sigma = \frac{8\pi}{3} \frac{e^4}{m^2 c^4} \lambda^{-4}$$

Electron scattering cross section
(fundamental)

Incoherent Scatter Concepts Are Older Than You Think

Without worrying about noise:

Rayleigh scattering $\propto \lambda^4$ [why is the sky blue?]

Incoherent scatter independent of wavelength
[but it's weak]

*Remarques sur la diffusion de la
lumière et des ondes hertziennes
par les électrons libres*

Incoherent Scatter concept!

C. Fabry
1928



Charles Fabry
1867-1945

For luminous radiations whose wavelength is very small, there is no phase relation between the elementary waves sent out by the different electrons of even a small volume and it is the intensities which add up. Thus, if a certain volume contains a total number of electrons n , then the power that it scatters is that transmitted by an area $S = n\sigma$. With the degrees of ionization that can actually exist, the scattering of light by electrons is always very slight. That is why it plays no appreciable role in the production of light in the diurnal sky *.

First Incoherent Scatter Radar

- W. E. Gordon of Cornell is credited with the idea for ISR.
- *“Gordon (1958) has recently pointed out that scattering of radio waves from an ionized gas in thermal equilibrium may be detected by a powerful radar.”* (Fejer, 1960)
- Gordon proposed the construction of the Arecibo Ionospheric Observatory for this very purpose (NOT for radio astronomy as the primary application)

~40 megawatt-acres



- 1000' Diameter Spherical Reflector
 - 62 dB Gain
- 430 MHz line feed 500' above dish
- Gregorian feed
- Steerable by moving feed.

Incoherent Scattering of Radio Waves by Free Electrons with Applications to Space Exploration by Radar*

W. E. GORDON†, MEMBER, IRE

INTRODUCTION

FREE electrons in an ionized medium scatter radio waves incoherently so weakly that the power scattered has previously not been seriously considered. The calculations that follow show that this incoherent scattering, while weak, is detectable with a powerful radar. The radar, with components each representing the best of the present state of the art, is capable of:

- 1) measuring electron density and electron temperature as a function of height and time at all levels in the earth's ionosphere and to heights of one or more earth's radii;
- 2) measuring auroral ionization;
- 3) detecting transient streams of charged particles coming from outer space; and
- 4) exploring the existence of a ring current.

* Original manuscript received by the IRE, June 11, 1958; revised manuscript received, August 25, 1958. The research reported in this paper was sponsored by Wright Air Dev. Ctr., Wright-Patterson Air Force Base, O., under Contract No. AF 33(616)-5547 with Cornell Univ.

† School of Elec. Eng., Cornell Univ., Ithaca, N. Y.



First Incoherent-Scatter Radar

- **K.L. Bowles [Cornell PhD 1955]**, Observations of vertical incidence scatter from the ionosphere at 41 Mc/sec. *Physical Review Letters* 1958:

“The possibility that incoherent scattering from electrons in the ionosphere, vibrating independently, might be observed by radar techniques has apparently been considered by many workers although seldom seriously because of the enormous sensitivity required...”

First Incoherent-Scatter Radar

...Gordon (W.E. Gordon from Cornell) recalled this possibility to the writer [spring 1958; D. T. Farley] while remarking that he hoped soon to have a radar sensitive enough to observe electron scatter in addition to various astronomical objects..."

Bowles executed the idea - hooked up a large transmitter to a dipole antenna array in Long Branch Ill., took a few measurements.

Gordon presenting on same day at October 21, 1958 Penn State URSI meeting:

"...And then I want to tell you about a telephone call that I just had."

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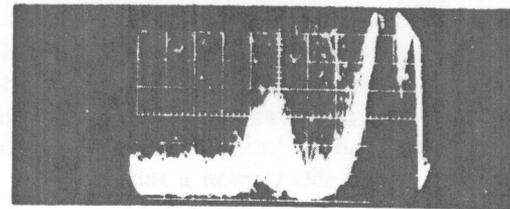
PHYSICAL REVIEW LETTERS

DECEMBER 15, 1958

Table I. Parameters of radar equipment used.

Operating frequency	40.92 Mc/sec
Peak pulse power	$(4 \text{ to } 6) \times 10^6$ watts
Pulse duration	$(50 \text{ to } 150) \times 10^{-6}$ sec
Average power	4×10^4 watts maximum
Receiver bandwidth	10, 15, or 30 kc/sec
Antenna cross section	116×140 meters (1024 half-wave elements in phase above ground)
Antenna polarization	north-south
Calculated antenna gain	~ 35 decibels/isotropic

~6 week setup time



Oscilloscope + camera + ~4 sec exposure
(10 dB integration)

FIG. 2. Pulse with 30 kc/sec bandwidth
30 kc.

Incoherent Scattering Detectability

Bowles' results found approximately the expected amount of power scattered from the electrons (scattering is proportional to charge to mass ratio - electrons scatter the energy).

BUT: his detection with a 20 megawatt-acre system at 41 MHz (high cosmic noise background; should be marginal) implies a spectral width 100x narrower than expected – almost as if the much heavier (and slower) ions were controlling the scattering spectral width.

In fact, they do.

Calculating the fluctuation spectrum

We need to find a way to calculate electron plasma fluctuations at a given wavelength and frequency - use statistical mechanics:

Equipartition of kT of energy between potential, kinetic energy; normal mode = 2 traveling waves with equally divided energy

$$\langle KE \rangle_{wave} = \frac{1}{4} k_B T = \frac{1}{2} N_0 m \langle |v(\vec{k}, \omega)|^2 \rangle$$


$$\langle |v(\vec{k}, \omega)|^2 \rangle = \frac{1}{2} \frac{k_B T}{N_0 m}$$

Add the continuity equation:

$$\frac{\partial N}{\partial t} + \nabla \cdot (N \vec{v}) = 0$$

Calculating the fluctuation spectrum

$$\left\langle \left| \Delta N(\vec{k}, \omega) \right|^2 \right\rangle = \frac{1}{2} N_0 \frac{k_B T}{m} \left(\frac{k^2}{\omega^2} \right)$$

Insert plasma dispersion relation here

We need the full dispersion relation expression.

Vlasov Equation

Boltzmann description of a gas:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\mathbf{p}}{m} + \frac{\partial f}{\partial \mathbf{p}} \cdot \mathbf{F} = \frac{\partial f}{\partial t} \Big|_{\text{coll}} .$$

Long range Coulomb forces not well described by the “molecular chaos” collision term (misses collective behaviors).

Start with collisionless Boltzmann, and adapt to plasma EM interaction forces:

$$\frac{\partial f_e}{\partial t} + \vec{v}_e \cdot \nabla f_e - e \left(\vec{E} + \frac{1}{c} (\vec{v} \times \vec{B}) \right) \cdot \frac{\partial f_e}{\partial \vec{p}} = 0 \quad \text{Electron, ion distr fns}$$

$$f_e(\vec{r}, \vec{p}, t)$$

$$\frac{\partial f_i}{\partial t} + \vec{v}_i \cdot \nabla f_i + e \left(\vec{E} + \frac{1}{c} (\vec{v} \times \vec{B}) \right) \cdot \frac{\partial f_i}{\partial \vec{p}} = 0 \quad f_i(\vec{r}, \vec{p}, t)$$

$$\nabla \times \vec{B} = \frac{4\pi \vec{j}}{c} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}, \quad \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{E} = 4\pi \rho, \quad \nabla \cdot \vec{B} = 0$$

$$\rho = e \int (f_i - f_e) d^3 \vec{p}, \quad \vec{j} = e \int (f_i - f_e) \vec{v} d^3 \vec{p}, \quad \vec{v}_\alpha = \frac{\vec{p}/m_\alpha}{(1 + p^2/(m_\alpha c)^2)^{1/2}}$$

Fluid Reductions: Moment Equations, Approximations

Integrate Vlasov over all velocity space:

$$\frac{\partial}{\partial t} n + \nabla \cdot (n \mathbf{u}) = 0$$

Continuity equation

Combining Vlasov with Lorentz equation:

$$m \frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

yields

$$m \frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

Momentum equation

Loses information on distribution function momentum space.

For analysis, often assume linearized, plane wave, small perturbations:

$$v_i = v_{i,0} + \Delta v_i \quad n_e = n_{e0} + \Delta n_e$$

Plasma Resonance Mode: Langmuir Oscillations

Assume electrostatic equations and no thermal motions - also, 1D:

$$mn_e \left[\frac{\partial \vec{v}_e}{\partial t} + (\vec{v}_e \cdot \nabla) \vec{v}_e \right] = -en_e \vec{E}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{v}_e) = 0$$

$$\epsilon_0 \nabla \cdot \vec{E} = e(n_i - n_e)$$

Combine, linearize, keep 1st order terms, assume plane waves:

$$-im\omega \Delta v = -i \frac{n_0 e^2}{\epsilon_0 \omega} \Delta v$$

Plasma Resonance Mode: Langmuir Oscillations

Plasma oscillations (Langmuir oscillations):

$$\omega_p = \left(\frac{n_0 e^2}{\epsilon_0 m} \right)^{1/2}$$

NB: group velocity = 0 so no information exchange in plasma.

Add electron thermal effects (now information is transmitted) to motion equation through an electron pressure gradient $-\nabla p_e$:

$$-\nabla p_e = 3k_B T_e \nabla(n_0 + \Delta n)$$

Linearize as before:

$$\omega^2 = \omega_p^2 + \frac{3}{2} k^2 v_{th}^2 \quad v_{th}^2 = 2k_B T_e / m_e$$

Irving Langmuir (1881 - 1957)



When Langmuir arrived at the Laboratory, the director, Willis R. Whitney, told him to look around and see if there was anything he would like to “play with.” Whitney would often ask him, “Are you having any fun today?” One day, after three years of apparently unproductive research, Langmuir answered, “I’m having a lot of fun, but I really don’t know what good this is to the General Electric Company.” Whitney replied. “That’s not your worry. That’s mine.”

Plasma Resonance Mode: Ion Acoustic Oscillations

Use fluid approximation for Vlasov equation (moment reductions):

$$mn \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = qn(\vec{E} + \vec{v} \times \vec{B}) - \vec{\nabla} p$$

Detour: neglect EM forces, add viscosity: back to Navier-Stokes

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\vec{\nabla} p + \rho \gamma \nabla^2 \vec{v}$$

Neglect viscosity: what is the characteristic dispersion relation?

$$\frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}$$

Plasma Resonance Mode: Ion Acoustic Oscillations

$$-\vec{\nabla} p = -\gamma \frac{p}{\rho} \nabla \rho$$

Add continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Linearize, solve:

$$\frac{\omega}{k} = \left(\frac{\gamma p_0}{\rho_0} \right)^{\frac{1}{2}} = \left(\frac{\gamma k_B T}{m} \right) = C_s$$

Normal pressure waves in a fluid: adiabatic compression and decompression of fluid particles.

Plasma Resonance Mode: Ion Acoustic Oscillations

Back to plasmas. Use quasineutrality (i.e. no Poisson equation), and add back E field - electrostatics:

$$m_i n \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = en \vec{E} - \vec{\nabla} p = -en \nabla \phi - \gamma k_B T_i \nabla n$$

Assume complete Boltzmann distribution for electrons:

$$n_e \equiv n = n_0 \left(1 + \frac{e\phi}{k_B T_e} + \dots \right)$$

So electron density perturbations:

$$\Delta n = n_0 \frac{e\phi}{k_B T_e}$$

Plasma Resonance Mode: Ion Acoustic Oscillations

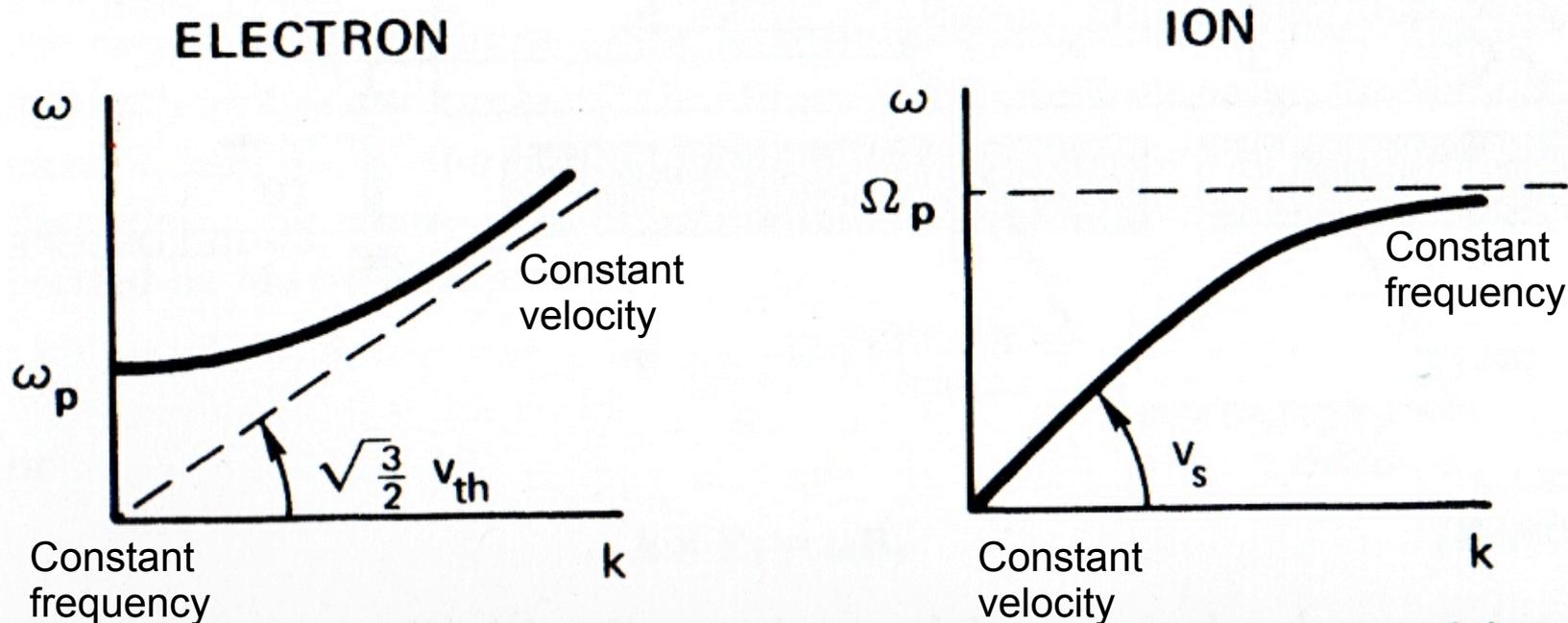
Add ion continuity equation, linearize, solve:

$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = V_s$$

These are ion-acoustic or ion pressure waves. Restoring force is the coupling electric field.

(Note no gamma term for electrons since they are isothermal, but ions are slow and suffer 1D compressions so their gamma term = 3)

Electron and ion waves: Dispersion relations



Comparison of the dispersion curves for electron plasma waves and ion acoustic waves.

Calculating the fluctuation spectrum: Review

$$\left\langle \left| \Delta N(\vec{k}, \omega) \right|^2 \right\rangle = \frac{1}{2} N_0 \frac{k_B T}{m} \left(\frac{k^2}{\omega^2} \right)$$

Insert plasma dispersion relation here

We need the full dispersion relation expression.

This is not a plasma waves course so we won't derive it, but the two most important modes are:

1) *Ion-acoustic fluctuations* [sound waves in plasma]

$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = V_s$$

NB: ordinary acoustic waves:
adiabatic compression /
decompression of fluid particles.

Ion-acoustic fluctuations:
restoring force = electromagnetic

Calculating the fluctuation spectrum

$$\left\langle \left| \Delta N(\vec{k}, \omega) \right|^2 \right\rangle = \frac{1}{2} N_0 \frac{k_B T}{m} \left(\frac{k^2}{\omega^2} \right)$$

Insert plasma dispersion relation here

We need the full dispersion relation expression.

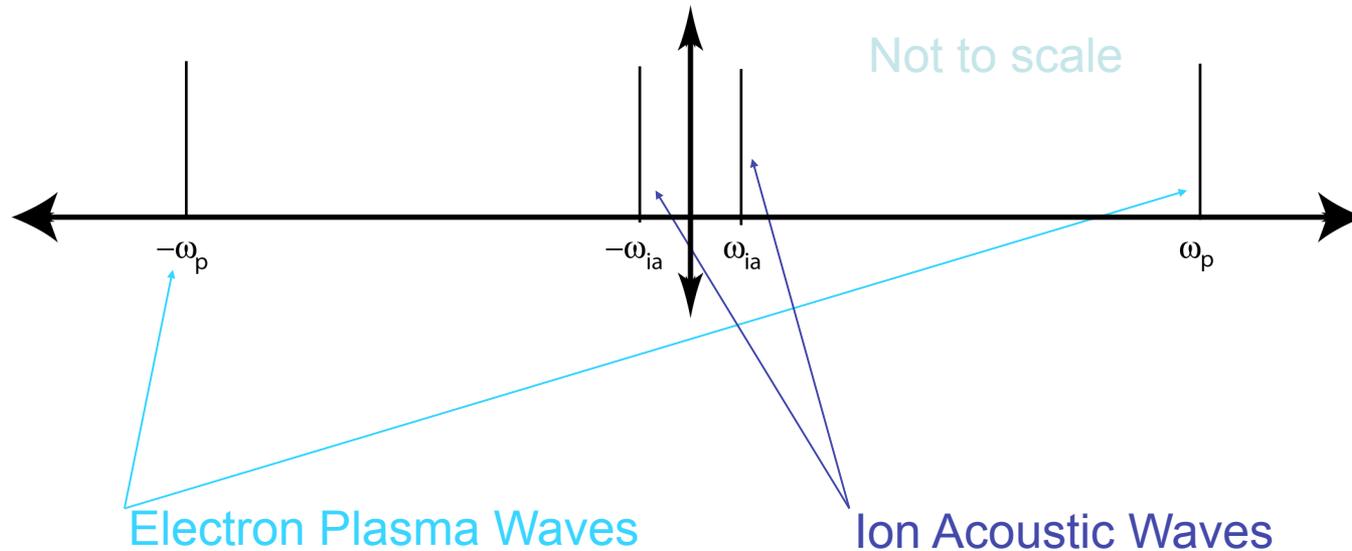
This is not a plasma waves course so we won't derive it, but the two most important modes are:

2) *Langmuir oscillations* (Plasma oscillations):

$$\omega^2 = \omega_p^2 + \frac{3}{2} k^2 v_{th}^2 \quad v_{th}^2 = 2k_B T_e / m_e$$

Akin to Brunt-Väisälä oscillations in fluid (parcel in presence of density gradient) - here, electrostatic field is restoring force, and electron pressure gradient transmits information

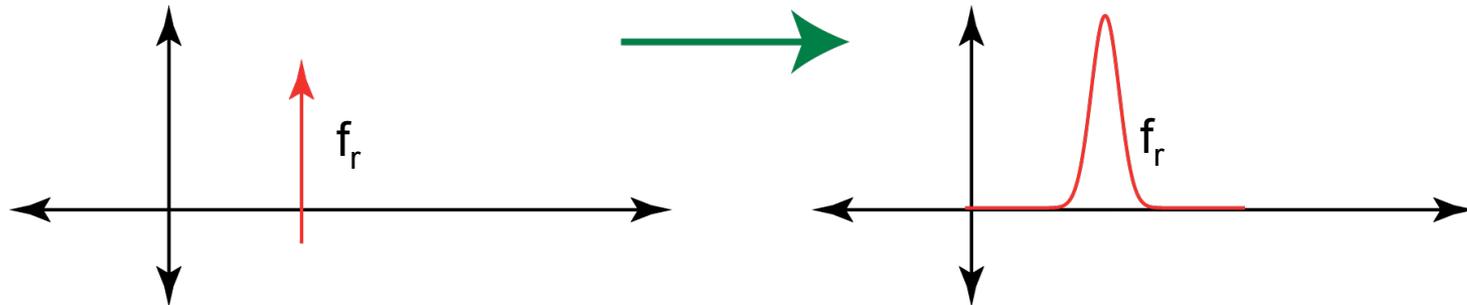
Wave Spectrum



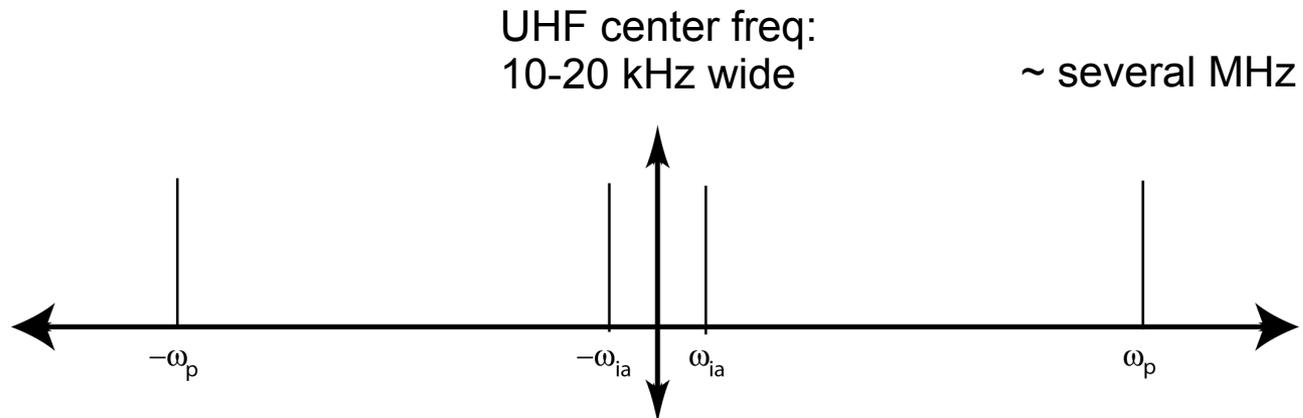
Plasma parameters fluctuate with the waves (density, velocity, etc)

Damped resonance

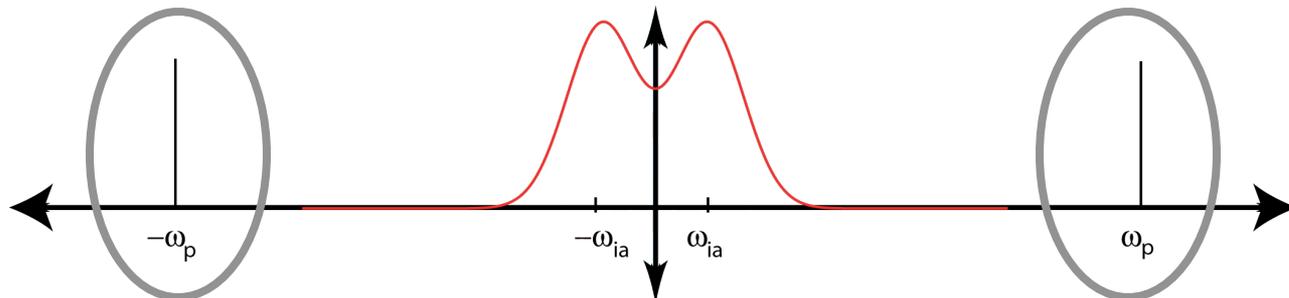
- Waves in a plasma are resonances.
- Damped resonances are not sharp
 - Example – Q of a resonant circuit.
- IS: Thermal ions have motions close to ion-acoustic speed (Landau damping – “surfing”; locked to I-A waves)



Wave Spectrum (ISR Spectrum)



Why aren't the Langmuir (plasma) waves damped?
Electron thermal velocity ~ 125 km/s but plasma wave frequency \sim several MHz –
Not much interaction and not much damping.



Nyquist-Johnson thermal noise (1928): Always present

JULY, 1928

PHYSICAL REVIEW

VOLUME 32

THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

By H. NYQUIST

ABSTRACT

The electromotive force due to thermal agitation in conductors is calculated by means of principles in thermodynamics and statistical mechanics. The results obtained agree with results obtained experimentally.



H. Nyquist 1889-1976
(born Nilsby, Sweden)

“Bert” Johnson 1887-1970
(born Gothenburg, Sweden)



Nyquist-Johnson: The Motivation



Nyquist-Johnson thermal noise (1928)

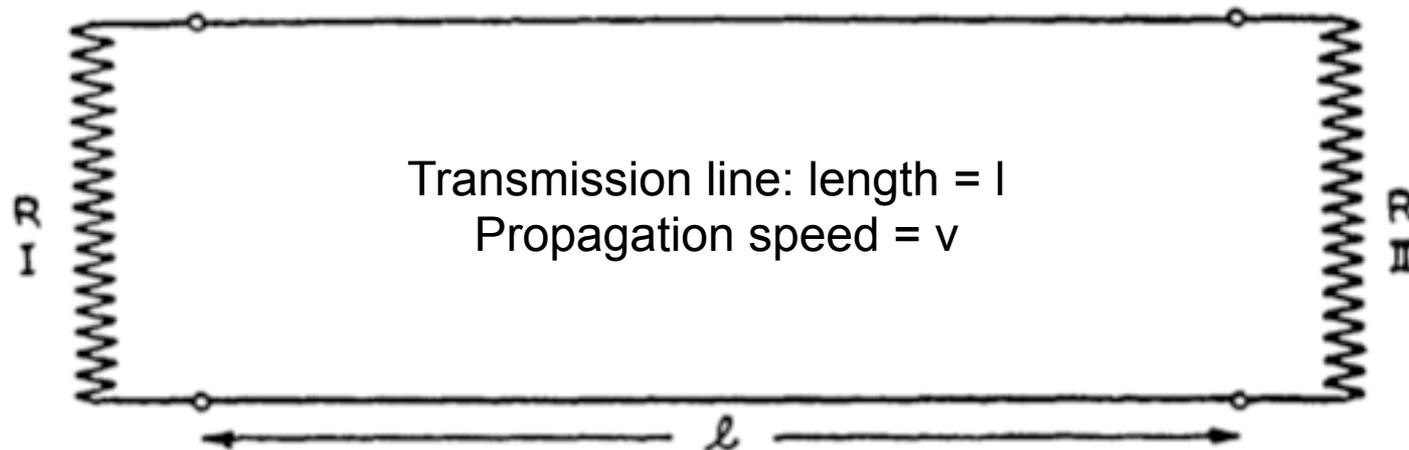


Fig. 3.

Natural transmission line resonance frequencies = $v/2l, 2v/2l, 3v/2l, \dots$

In general, degrees of freedom = $2 l d\nu / v$ for a frequency interval $d\nu$

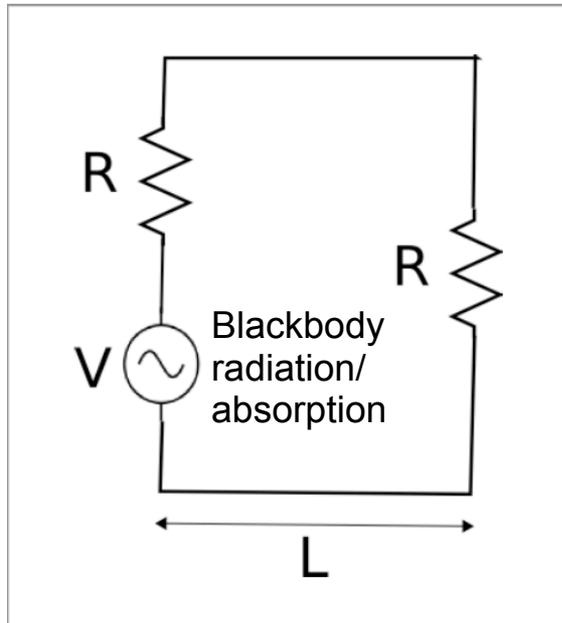
Each degree of freedom has kT of thermal energy.

2 conductors (R above), each contributing energy to line.

The energy gets transferred to the line during its travel time l / v :

$$P = k_B T \Delta f$$

Nyquist-Johnson thermal noise (1928)



$$P_{absorbed} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{h|\omega|}{e^{\frac{h|\omega|}{k_B T}} - 1}$$

Absorbed = emitted power
(Thermal equilibrium)

$h|\omega|$: Planck's law

$$N_n = \frac{g_n \leftarrow 2}{\exp\left[\frac{\epsilon_n}{kT}\right] - 1}$$

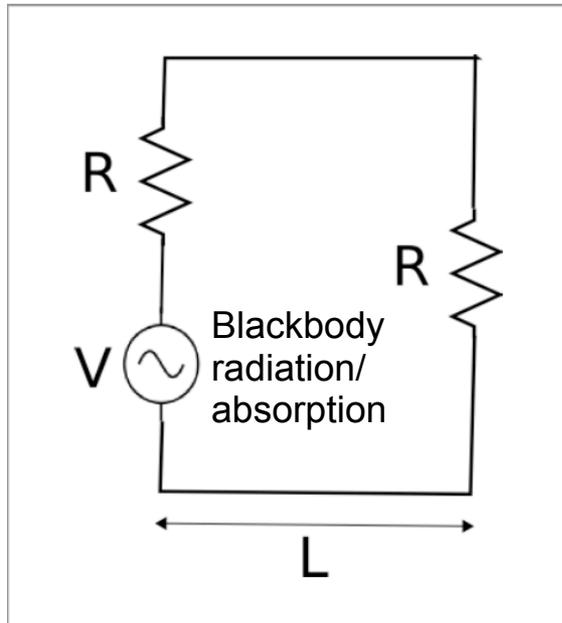
Bose-Einstein statistics:
(2 = photons have 2 polarizations)

Solve 1D wave equation in the circuit:

Power flow density

$$P(\omega) = \frac{1}{2} v U(\omega) = \frac{1}{2\pi} \frac{\hbar\omega}{\exp\left[\frac{\hbar\omega}{kT}\right] - 1} d\omega$$

Nyquist-Johnson thermal noise (1928)



$$P(\omega) = \frac{1}{2} v U(\omega) = \frac{1}{2\pi} \frac{\hbar \omega}{\exp[\hbar \omega / kT] - 1} d\omega$$

Change to frequency:

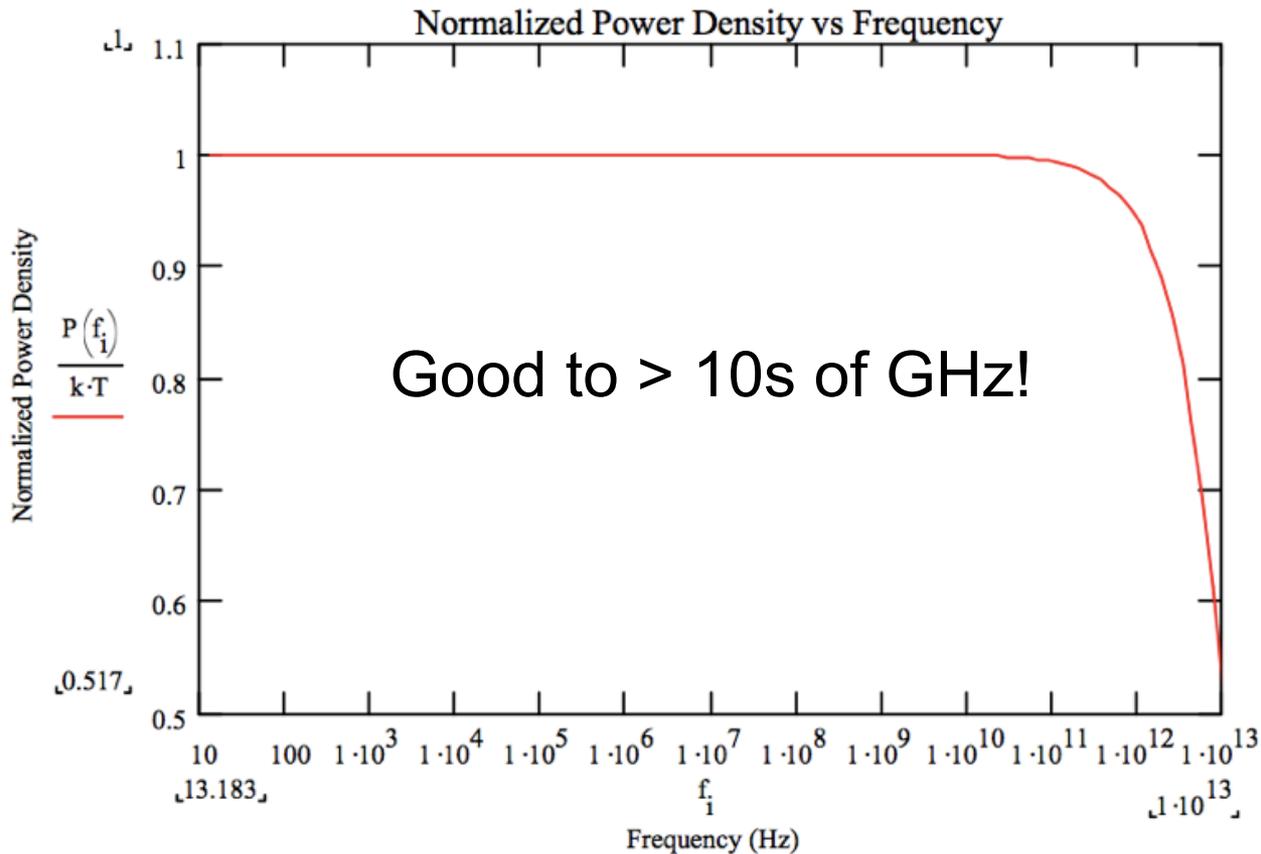
$$P(f) = \frac{hf}{\exp[hf/kT] - 1} df$$

Classical limit ($kT \ll hf$): Taylor expand -

$$\hat{P}(f) = kT \cdot df$$

Thermal noise power independent of resistance
(but thermal voltage in the circuit does depend on R)

Nyquist-Johnson thermal noise (1928)



(Clay Turner, Wireless Solutions, 2007)

Nyquist generalization: Fluctuation-dissipation theorem

Powerful tool in statistical physics.
Fundamental assumption:

If our system is in thermal equilibrium (*),
The response we see to a small applied force is the same as its
response to a spontaneous fluctuation (i.e. statistical fluctuations in
equilibrium).

Nyquist was using this principle for the derivation we have just seen.

It can be proven and generalized.

(*) There are also modifications for it to work in cases where thermal
equilibrium does not exist - e.g. multiple species or populations, each
with its own thermal distribution function.

Nyquist generalization: Fluctuation-dissipation theorem

Callen and Welton [1951]:

Any linear dissipative system whose components are in thermal equilibrium will exhibit thermally driven fluctuations having power spectra which can be derived by applying Nyquist noise principles to an equivalent circuit model system.

We therefore need to derive such a model for thermal plasma.

Ampere's law:
$$-j\vec{k} \times \vec{H} = \vec{J} + j\omega\epsilon_0\vec{E}$$

becomes
(in k dir)
$$0 = (\sigma_i + \sigma_e)E + \frac{\omega}{k}e(n_{th,i} - n_{th,e}) + j\omega\epsilon_0E$$

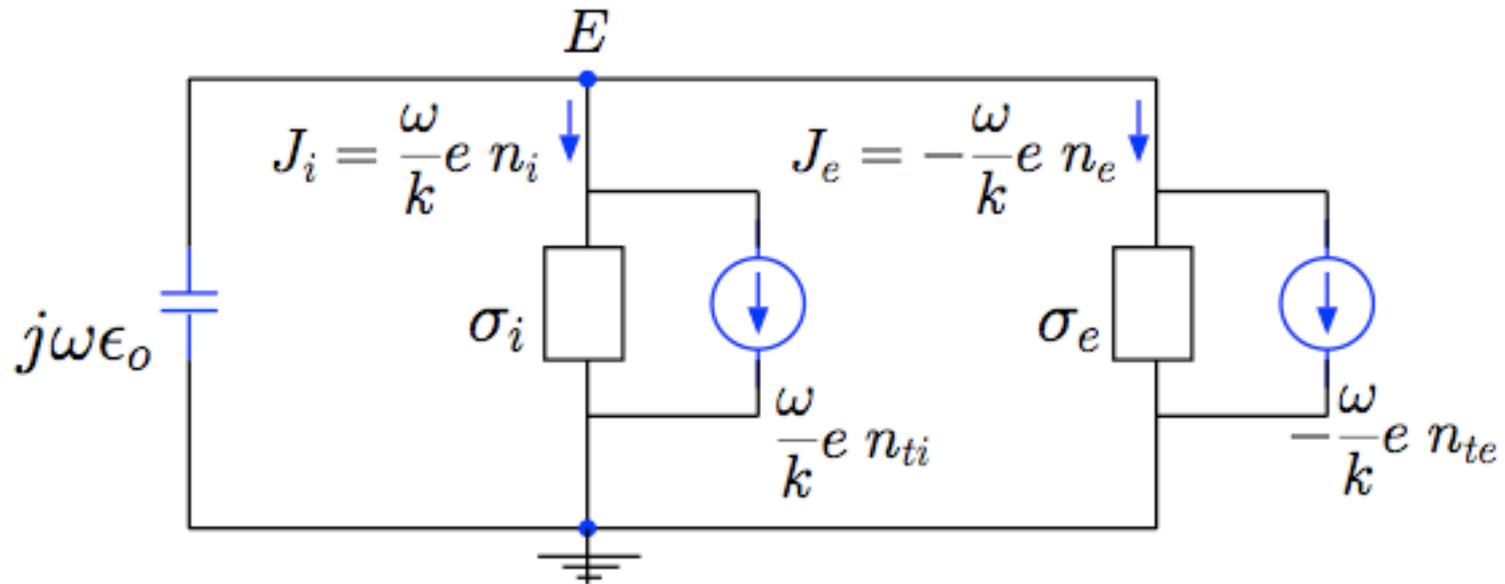
Conduction currents
(particle motion deviations)

Thermal random currents
(and Poisson's law for
overall plasma neutrality)

Displacement current

Fluctuation-dissipation: Plasma Circuit Model

$$0 = (\sigma_i + \sigma_e)E + \frac{\omega}{k}e(n_{th,i} - n_{th,e}) + j\omega\epsilon_0 E$$



Solve for E and use in current continuity to get density fluctuations

$$-\frac{\omega}{k}en_e = E\sigma_e - \frac{\omega}{k}en_{th,e}$$

Electron density fluctuation spectrum

Solution yields

$$\left\langle \left| n_e(\vec{k}, \omega) \right|^2 \right\rangle = \frac{|j\omega\epsilon_0 + \sigma_i|^2 \left\langle \left| n_{th,e}(\vec{k}, \omega) \right|^2 \right\rangle}{|j\omega\epsilon_0 + \sigma_e + \sigma_i|^2} \quad \text{Electron line}$$
$$+ \frac{|\sigma_e|^2 \left\langle \left| n_{th,i}(\vec{k}, \omega) \right|^2 \right\rangle}{|j\omega\epsilon_0 + \sigma_e + \sigma_i|^2} \quad \text{Ion line}$$

Beyond this point, use plasma kinetic theory to derive ion and electron conductivities in a thermal plasma.

cf. Dougherty and Farley (1960) and “Farley series”

Compare to other approaches: Fejer (1960), Salpeter (1960), Hagfors (1961)

Electron density fluctuation spectrum

Use electron force/response concept and solve for electron and ion admittances y_e, y_i (analogous to resistive dissipation). Arrive at spectral expression

$$\sigma_0(\omega_o + \omega)d\omega = N_0 r_e^2 \operatorname{Re} \left\{ \frac{y_e(y_i + jk^2 \lambda_{de}^2)}{y_e + y_i + jk^2 \lambda_{de}^2} \frac{d\omega}{\pi\omega} \right\}$$

- Short wavelength limit ($k^2 \lambda_{de}^2 \gg 1$): pure e^- scatter
- Long wavelength limit: RHS $\rightarrow y_e y_i / (y_e + y_i)$: damped ion-acoustic resonances
- Near plasma frequency: $y_e + y_i + jk^2 \lambda_{de}^2 \rightarrow 0$: plasma lines

Incoherent Scatter Spectral Dependence

Spectral response can be evaluated using these frameworks for:

- Thermal inequality $T_e \neq T_i$: decreases Landau damping
- Ion-neutral collisions ν_{in} : narrows spectrum
- Background magnetic field B_0 : makes electrons heavier

$$m_e \rightarrow m_e^* = \frac{m_e}{\cos^2 \alpha}$$

Also, ion gyro-resonance (mass-dependent).

Incoherent Scatter Spectral Dependence

- Ion mixtures: $\frac{T_e}{T_i} y_i \rightarrow \sum_j \frac{T_e}{T_j} \frac{N_j}{N_0} y_j(m_j, T_j)$
- Unequal ion temperatures
- Particle drifts: $\omega \rightarrow \omega - \vec{k} \cdot \vec{v}_{de}$
- Plasma line measurements ($[e^-]$, T_e , $v_{||}$)
- Photoelectron heating, non-Maxwellian plasmas
- Faraday rotation effects (equator, low TX freq)

Things can get hairy. For example, magnetic field evaluation requires Gordeyev integral:

$$\int e^{j(\theta - j\phi)t - \frac{\sin^2 \alpha}{\phi^2} \sin^2(\frac{\phi t}{2}) - \frac{t^2}{4} \cos^2 \alpha} dt$$

IS Spectral Shape Demonstration

(See IS Spectrum Java applet on “ISR Demonstration” page)

Measurement Statistics

$E_s(t)$ and $\therefore V_s(t)$ are Gaussian random variables
(Central Limit Theorem):

$$\begin{aligned}V_s(t) &= V_1 &&= x_1 + jx_2 \\V_s(t + \tau) &= V_2 &&= x_3 + jx_4\end{aligned}$$

We desire ensemble averages of 2nd moments
(correlations):

$$\langle V_1 V_2^* \rangle = \langle (x_1 + jx_2)(x_3 + jx_4)^* \rangle = S\rho(\tau)$$

where S is signal power, and IS theory gives medium
correlation

$$\rho(\tau) = \rho_R(\tau) + j\rho_I(\tau)$$

Measurement Statistics

In general, we define an estimator to approximate true ensemble average - e.g.

$$\hat{S} = \frac{1}{K} \sum_{i=1}^K V_i V_i^*$$

might be power estimator for true $S = \langle V_1 V_1^* \rangle$.
Each estimator will have an associated bias and variance, e.g.

$$\text{bias} = \langle \hat{S} \rangle$$

$$\text{variance}(\hat{S} - S) = \langle (\hat{S} - S)^2 \rangle$$

Power Estimation

For total scattered power, use

$$\hat{S} = \frac{1}{K} \sum_{i=1}^K V_i V_i^*$$

$$\text{Bias : } \hat{S} = S$$

$$\text{Variance : } \langle (\hat{S} - S)^2 \rangle = \frac{S^2}{K} = \delta_S^2$$

$$\text{RMS frac error : } \frac{\delta_S}{S} = \frac{1}{\sqrt{K}}$$

10,000 samples needed for 1% accuracy.

Power Estimation: Noise Effects

Add noise (Gaussian RV with different 2nd moment).
Use estimator

$$\hat{S} = S + \hat{N} - \hat{N}$$

$$\text{Bias : } \hat{S} = S$$

$$\text{Variance : } \delta_S^2 \sim \frac{(S + N)^2}{K_{S+N}}$$

$$\text{RMS frac error : } \frac{\delta_S}{S} \sim \frac{S + N}{S} \frac{1}{\sqrt{K_{S+N}}}$$

ACF Estimation

We want $\langle V(t)V^*(t + \tau) \rangle = \langle V_1 V_2^* \rangle = S\rho(\tau)$. A popular estimator is:

$$\hat{\rho} = \frac{\frac{1}{K} \sum_{i=1}^K V_{1i} V_{2i}^*}{\left[\frac{1}{K^2} \sum_{i=1}^K |V_{1i}|^2 \sum_{i=1}^K |V_{2i}|^2 \right]^{\frac{1}{2}}} = \frac{A}{B}$$

After linearizing and lots of details:

$$\text{Bias : } \hat{\rho} = \rho \left(1 - \frac{1}{4K} (1 - |\rho|^2) \right)$$

$$\text{Variance : } \delta_{\rho}^2 = \frac{1}{K} \left[1 - \frac{3}{2} |\rho|^2 + \frac{1}{2} |\rho|^4 \right]$$

ACF Estimation: Noise Effects

Effect of adding noise is to change the estimator:

$$\hat{\rho} = \frac{A_{S+N} - A_N}{B_{S+N} - B_N}$$

Details show that

$$\delta_{\hat{\rho}}^2 \sim \frac{1}{K} \left(\frac{S+N}{S} \right)^2 \left[1 + \frac{1}{2} |\rho_S|^2 \right]$$

Consequences:

- When SNR low, variance large
- Larger S is wasted statistically

IS Signal Chain: Statistical Demos with Real Data

(See IS Signal Chain on “ISR Demonstration” page)

Clutter Removal

Not all radar signals have the same correlation time. This can be an advantage in separating signals you want from signals that you don't want.

In particular, sometimes ground scatter from features such as mountains ends up at the same range delays as signals of interest – e.g. the E region. This radar clutter obscures the desired ionospheric signal and can be many orders of magnitude larger.

However, the clutter can have a much longer correlation time (many pulses) compared with the < 1 pulse typical of incoherent scatter. This can be exploited to subtract the clutter at the voltage level.

[Exercise/demo from the ISR Demonstration signal chain]