Scattering Model

Incident EM wave accelerates each charged particle it encounters.
These then re-radiate an EM wave.

For a single electron located at r = 0, the scattered field at a distance r_s :

scattered field
$$\left|\vec{E}_s(\vec{r_s}, t)\right| = \frac{e^2 \mu_0 \sin\delta}{4\pi R m_e} \left|\vec{E}_i(0, t')\right|$$
Incident field
$$= \frac{r_e}{R} \sin\delta \left|\vec{E}_i(0, t')\right|$$

$$r_e = \frac{e^2 \mu_0}{4\pi m_e}$$ Classical electron radius

$$t' = t - \frac{R}{c}$$ Delayed time

$\sin\delta$ Scattering angle

Scattering Model

Assume a volume filled with electron scatterers whose density is represented in space and time by

$$N(\vec{r}, t)$$

Illuminating this volume with an incident field from a transmitter location means that each electron contributes to the resulting scattered field, using Born approximation (each scatter is weak and does not affect others).

With geometrical considerations, scattered field at receiver location is now:

$$E_s(t) = r_e \sin\delta \; E_0 e^{j\omega_0 t} \int_{V_s} \frac{1}{r_s} N(\vec{r}, t') e^{-j(\vec{k_i} - \vec{k_s})\vec{r}} d^3\vec{r}$$

$$t' = t - \frac{r_i}{c}$$ Delayed time

Scattering Model

Assume densities have random spatial and temporal fluctuations about a background:

$$N(\vec{r}, t) \rightarrow N_0 + \Delta N(\vec{r}, t)$$

Further, assume backscatter (i.e. monostatic radar):

$$\vec{k} = 2\vec{k_i} \qquad\qquad r_i \equiv r_s = R$$

Then, scattered field reduces to:

$$E_s(t) \rightarrow \frac{r_e}{R} \sin\delta \; E_0 e^{j\omega_0 t} \underbrace{\int_{V_s} \Delta N(\vec{r}, t') e^{-j\vec{k}\cdot\vec{r}} d^3\vec{r}}_{\equiv \Delta N(\vec{k}, t')}$$

Scattering Model

Plasmas (ionosphere) are thermal gases and $\Delta N(\vec{r}, t)$ is a Gaussian random variable, so the Central Limit Theorem applies:

statistical average → $$\langle E_s(t) \rangle = \langle \Delta N(\vec{r}, t) \rangle = 0$$

It's much more useful to look at second order products – in other words, examine <u>temporal correlations</u> in the scattered field:

$$\langle E_s(t)\ E_s^*(t+\tau) \rangle \propto\ e^{-j\omega_0 \tau} \left\langle \Delta N(\vec{k}, t)\ \Delta N^*(\vec{k}, t+\tau) \right\rangle$$

Useful things to measure can now be defined.

Scattering: Measurable Quantities

Defining $C_s = \frac{r_e^2 E_0^2 \sin^2 \delta}{R^2} V_s$, then

Total scattered power

$$\left\langle |E_s(t)|^2 \right\rangle = C_s \left\langle |\Delta N(\vec{k})^2 \right\rangle$$

and Autocorrelation function (ACF):

$$\langle E_s(t) E_s^*(t+\tau) \rangle = C_s e^{-j\omega_0 \tau} \left\langle \Delta N(\vec{k}, t) \Delta N^*(\vec{k}, t+\tau) \right\rangle$$

or Power Spectrum:

$$\left\langle |E_s(\omega_0 + \omega)|^2 \right\rangle \propto C_s \left\langle \left| \Delta N(\vec{k}, w) \right|^2 \right\rangle$$

Incoherent Scattering: A Bragg Experiment

IS radar uses Bragg scattering: radar picks out one point in 3-D k space, and obtains characteristic spectral density at that spatial scale:

$$\vec{k} = \vec{k}_i - \vec{k}_s$$

Contrast this to a sounding rocket or satellite, which responds to an integral in k space perpendicular to its trajectory in the z direction:

$$\Delta N(\omega') = \int\int\int \Delta N(k_x, k_y, k_z; \omega)\ dk_x\ dk_y\ d\omega$$

$$k_z = -\frac{\omega'}{v_r}$$

Incoherent Scattering Model: Summary

Radar filters in k space:

$$\Delta N(\vec{r}, t) \rightarrow \Delta N(\vec{k}_r, t)$$

$$\Delta N(\vec{k}_r, t) \propto E_s(t)$$

Form ACF of $E_s(t)$ for each range, average, transform:

$$\langle E_s(t) E_s^*(t+\tau) \rangle \rightarrow \left\langle \left| \Delta N(\vec{k}, w) \right|^2 \right\rangle$$

Interpret latter in terms of the medium parameters.

Incoherent Scattering Detectability

Assume beam-filling F region plasma at 300 km altitude: $[e^-] \approx 10^{12}/m^3$

- Classical electron cross-section $\sigma_e = 10^{-28} m^2/e^-$
- Pulse length 10 km
- Beam cross-section 1 km (about Arecibo beamwidth)
- Total scattering cross-section $\sigma_{tot} \approx 10^{-6} m^2$

NB: total fraction of scattered power in target volume is 10^{-12} so Born approximation is good!

Incoherent Scattering Detectability

For fraction of power scattered actually received, assume isotropic scatter and a BIG 100 m diameter antenna:

$$f_{rec} = \frac{A_{rec}}{4\pi R^2} \approx \frac{(100\text{m})^2}{4\pi(300\text{km})^2}$$

$$f_{rec} \approx 10^{-8}$$

$$P_{rec}/P_{tx} \approx 10^{-20}$$

So a radar with a 1 MW transmitter receives 10^{-14} watts of incoherently scattered power.

Incoherent Scattering Detectability

What matters, though, is the signal to noise ratio:

$$P_{noise} = k_B T_{\text{eff}} BW$$

Typical effective temperatures $\sim$ 100 K at UHF frequencies (f_{Tx} = 430 MHz).

Assume bandwidth set by electron thermal velocity:

$$v_{e,thermal} \propto \sqrt{\frac{k_B T_e}{m_e}} \sim 10^5 m/s$$

$$BW \propto \frac{v_{e,thermal}}{c} f_{Tx}(2)(2)$$

$$\sim 500\text{kHz}$$

Incoherent Scattering Detectability

Finally,

$$P_{noise} \sim 10^{-15} W$$

so

$$S/N \sim 10$$

Not bad…

But you need a megawatt class transmitter and a huge antenna.

Fortunately, technology makes this possible in the mid 1950s.

First Incoherent Scatter Radar

- W. E. Gordon of Cornell is credited with the idea for ISR.
- *"Gordon (1958) has recently pointed out that scattering of radio waves from an ionized gas in thermal equilibrium may be detected by a powerful radar."* (Fejer, 1960)
- Gordon proposed the construction of the Arecibo Ionospheric Observatory for this very purpose (NOT for radio astronomy as the primary application)

~40 megawatt-acres

- 1000' Diameter Spherical Reflector
  - 62 dB Gain
- 430 MHz line feed 500' above dish
- Gregorian feed
- Steerable by moving feed.

Proceedings of the IRE, November 1958

Incoherent Scattering of Radio Waves by Free Electrons with Applications to Space Exploration by Radar*

W. E. GORDON†, MEMBER, IRE

Introduction

FREE electrons in an ionized medium scatter radio waves incoherently so weakly that the power scattered has previously not been seriously considered. The calculations that follow show that this incoherent scattering, while weak, is detectable with a powerful radar. The radar, with components each representing the best of the present state of the art, is capable of:

1) measuring electron density and electron temperature as a function of height and time at all levels in the earth's ionosphere and to heights of one or more earth's radii;
2) measuring auroral ionization;
3) detecting transient streams of charged particles coming from outer space; and
4) exploring the existence of a ring current.

* Original manuscript received by the IRE, June 11, 1958; revised manuscript received, August 25, 1958. The research reported in this paper was sponsored by Wright Air Dev. Ctr., Wright-Patterson Air Force Base, O., under Contract No. AF 33(616)-5547 with Cornell Univ.

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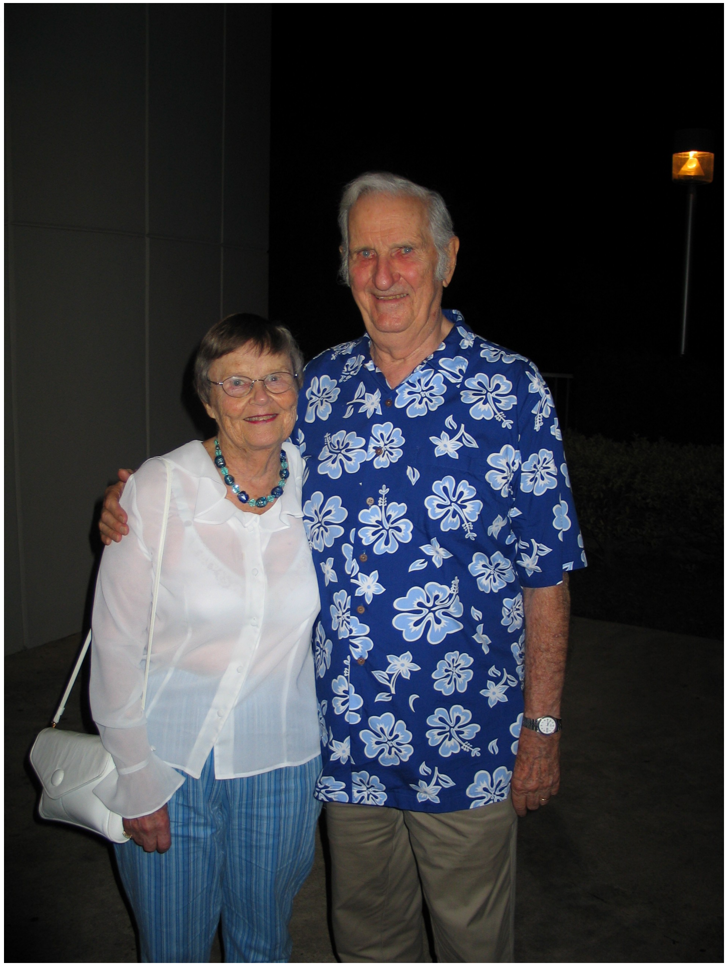


First Incoherent-Scatter Radar

- **K.L. Bowles [Cornell PhD 1955]**, Observations of vertical incidence scatter from the ionosphere at 41 Mc/sec. *Physical Review Letters* 1958:

"*The possibility that incoherent scattering from electrons in the ionosphere, vibrating independently, might be observed by radar techniques has apparently been considered by many workers although seldom seriously because of the enormous sensitivity required…*"

First Incoherent-Scatter Radar

…*Gordon (*W.E. Gordon from Cornell*) recalled this possibility to the writer* [spring 1958; D. T. Farley] *while remarking that he hoped soon to have a radar sensitive enough to observe electron scatter in addition to various astronomical objects…"*

Bowles executed the idea - hooked up a large transmitter to a dipole antenna array in Long Branch Ill., took a few measurements.

Gordon presenting on same day at October 21, 1958 Penn State URSI meeting:

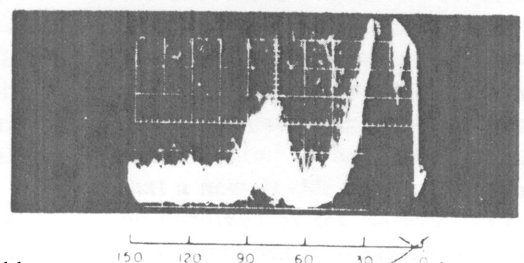
"…And then I want to tell you about a telephone call that I just had."

VOLUME 1, NUMBER 12 PHYSICAL REVIEW LETTERS DECEMBER 15, 1958

Table I. Parameters of radar equipment used.

Operating frequency	40.92 Mc/sec
Peak pulse power	(4 to 6) × 10^6 watts
Pulse duration	(50 to 150) × 10^{-6} sec
Average power	4 × 10^4 watts maximum
Receiver bandwidth	10, 15, or 30 kc/sec
Antenna cross section	116 × 140 meters (1024 half-wave elements in phase above ground)
Antenna polarization	north-south
Calculated antenna gain	~35 decibels/isotropic

~6 week setup time



Oscilloscope + camera + ~4 sec exposure (10 dB integration)

FIG. 2. Pulse [illegible] bandwidth 30 kc.

Incoherent Scattering Detectability

Bowles' results found approximately the expected amount of power scattered from the electrons (scattering is proportional to charge to mass ratio - electrons scatter the energy).

BUT: his detection with a 20 megawatt-acre system at 41 MHz (high cosmic noise background; should be marginal) implies a spectral width 100x narrower than expected – almost as if the much heavier (and slower) ions were controlling the scattering spectral width.

In fact, they do.

Incoherent Scatter Theoretical Approaches

Two main approaches to deriving IS spectrum from physical variables (density, temperature..):

- Dressed particles: use "test" particles, interacting by Debye clouds
- Plasma wave approach: study intrinsic resonant plasma modes, use Nyquist force/response calc to derive spectrum

Each has useful physical insights..

Dressed Particles

Insert a test ion in plasma - Debye shield forms. In thermal equilibrium, density change leads to electron (and ion) clouds around ion:

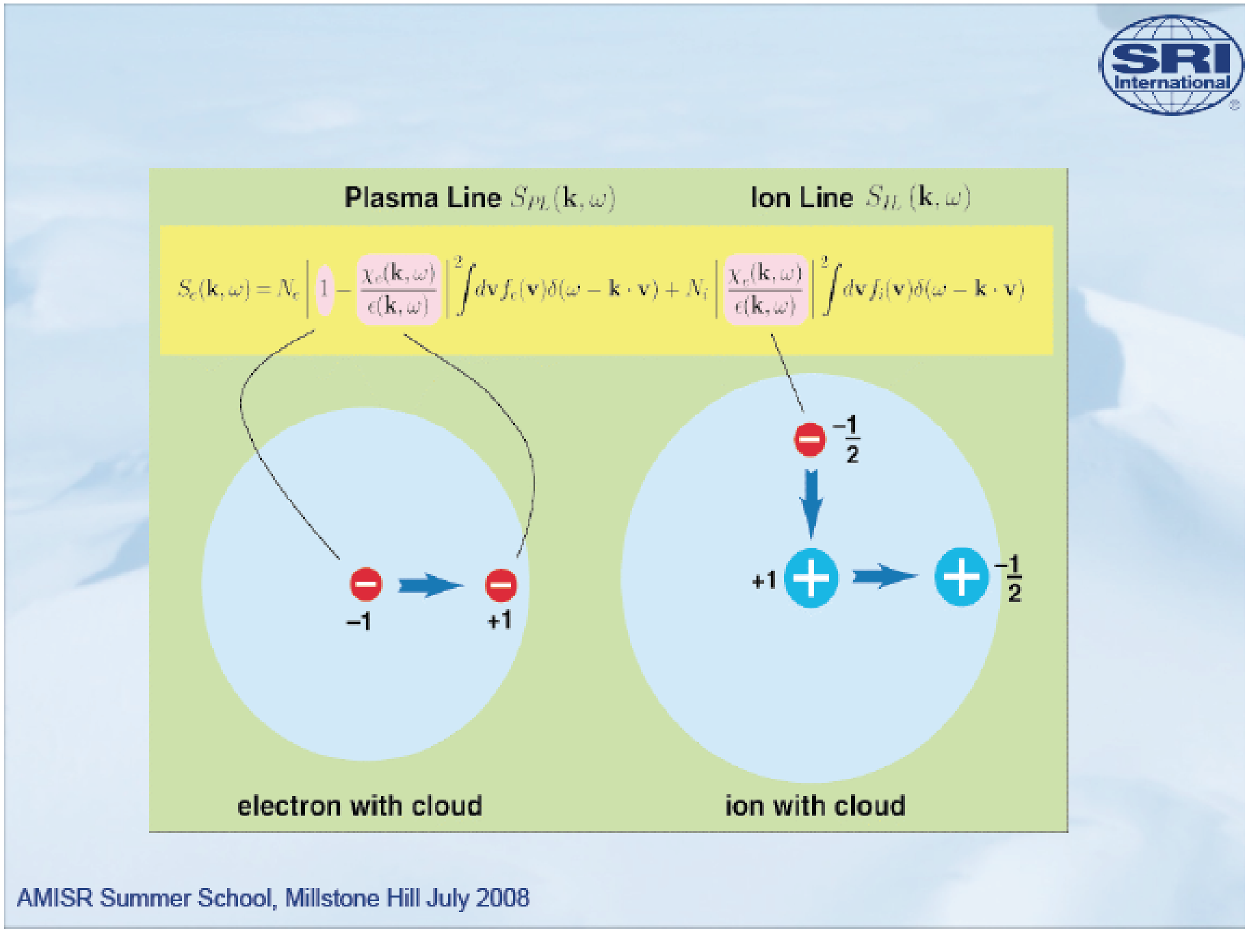
$$\Delta N_e = \frac{N_0}{1 + T_e/T_i} \rightarrow \frac{N_0}{2}$$

$$\Delta N_i = \frac{-N_0}{1 + T_i/T_e} \rightarrow \frac{-N_0}{2}$$

and we can scatter off electron cloud:

- Test ion, ion cloud: too heavy
- Test electron: electron depletion cloud cancels it

Dressed Particles



Dressed Particles

Relative to thermal velocity $\sqrt{\frac{k_B T_i}{m_i}}$:

$$\text{Slow ion}: \ \sigma_{radar} = \frac{4\pi r_e^2}{(1 + T_e/T_i)^2}$$

$$\text{Fast ion (no shielding by ions)}: \ \sigma_{radar} \rightarrow 4\pi r_e^2$$

Combined with thermal velocity spread, gives us classic double-hump IS spectrum.

Plasma Wave Approach

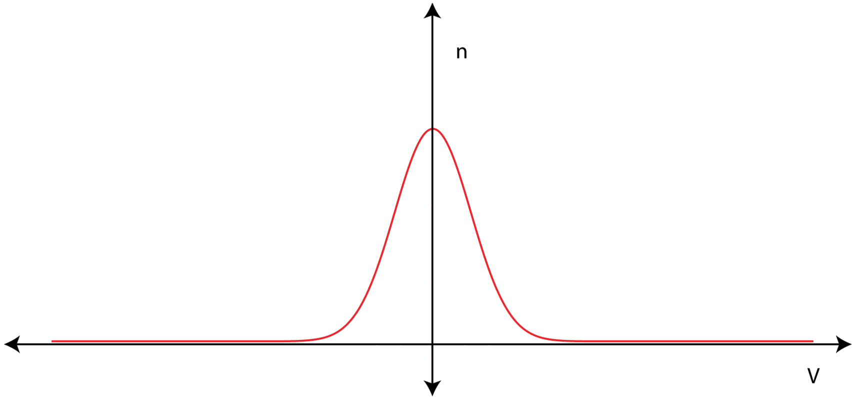
Treat plasma as a sum of characteristic resonant modes (from dispersion relation). Important ones:

- Plasma oscillation: $\omega = \omega_p \rightarrow$ Plasma lines
- Undamped ion-acoustic waves:
  $\omega^2 = k^2 c_s^2 \propto \frac{k_B(T_e+T_i)}{m_i} \rightarrow$ Ion lines

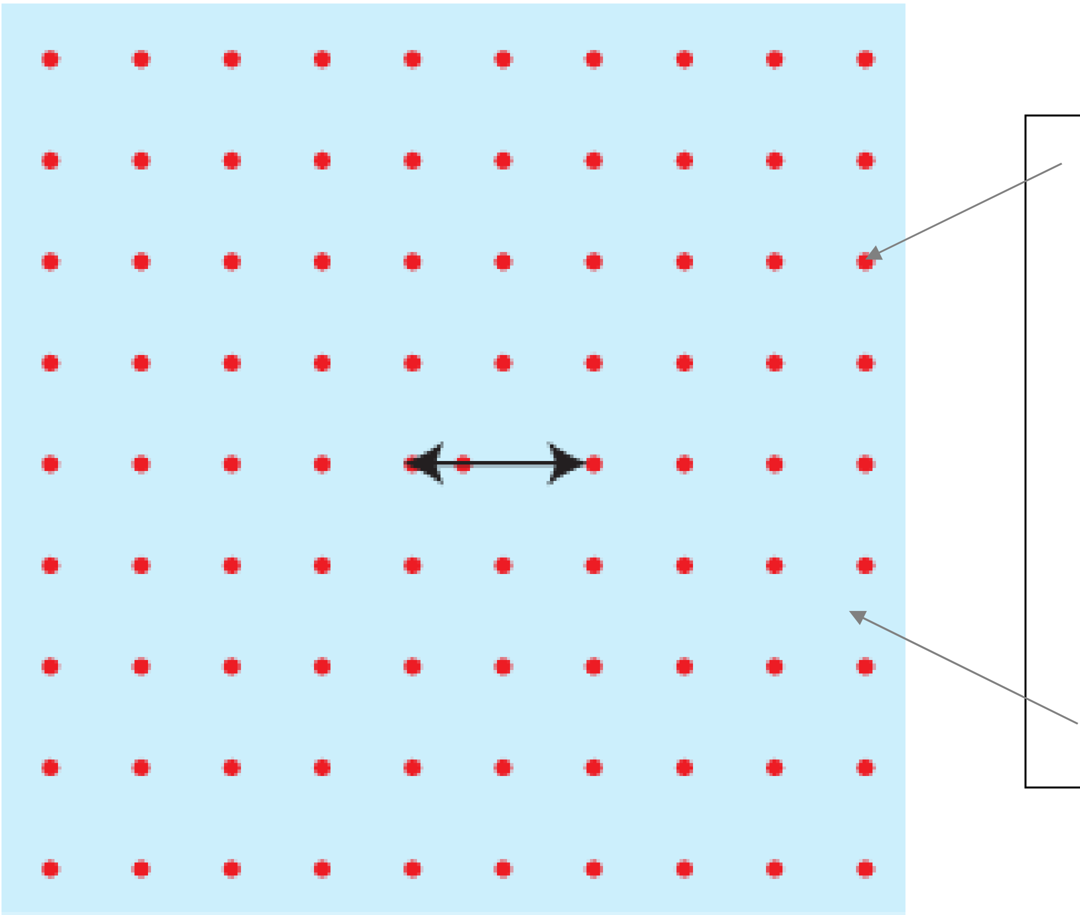
In practice, Landau damping broadens ion-acoustic modes considerably.

Density Fluctuations

- Thermal fluctuations in an ordinary collision dominated gas can be considered to be made up of sound waves.
- In a plasma, the fluctuations are ion-acoustic waves and electrostatic plasma (Langmuir) waves.
- The probability distributions for the wave modes and their spectrum can be derived by various means.



Ion Acoustic Waves

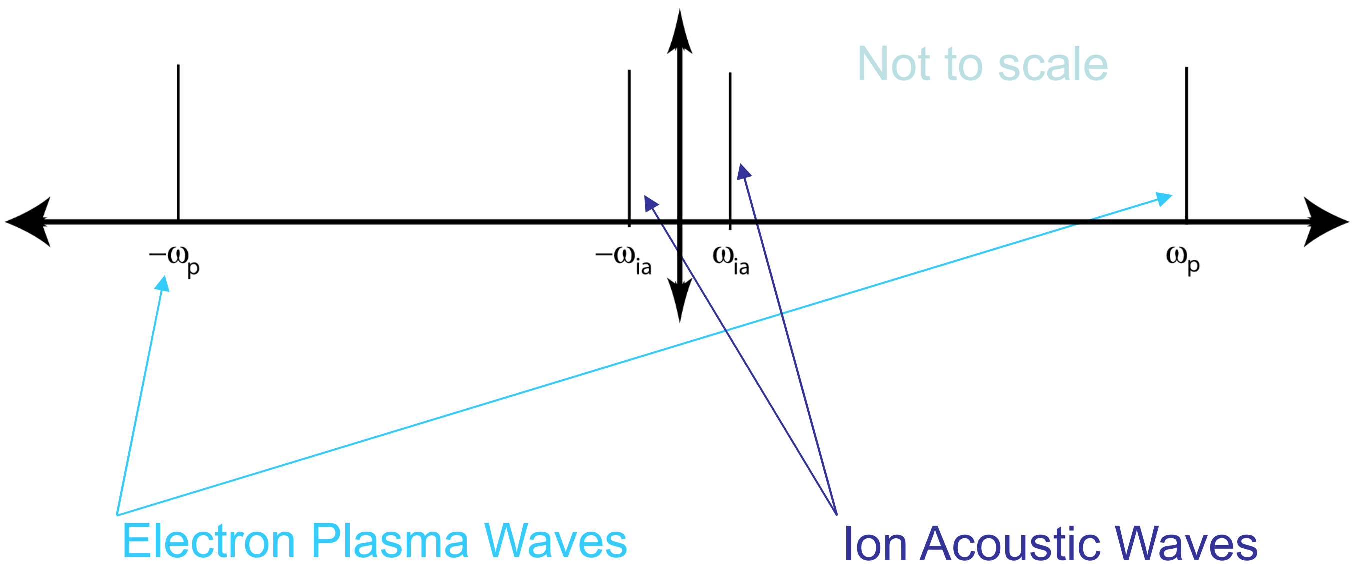


Thermal velocity

$$v_{TH} = \sqrt{\frac{2kT}{m}}$$

$$v_{THe} = \sqrt{\frac{m_i}{m_e}} v_{THi}$$

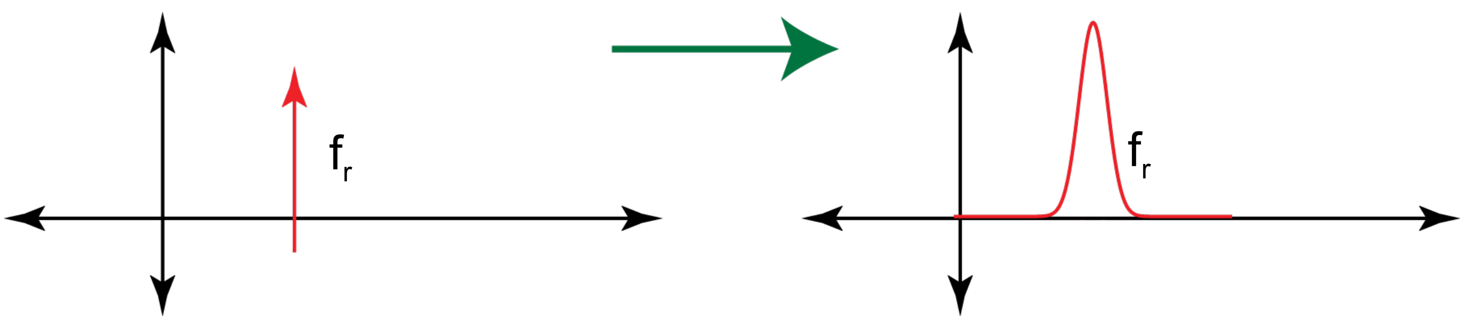
Wave Spectrum



Plasma parameters fluctuate with the waves (density, velocity, etc)

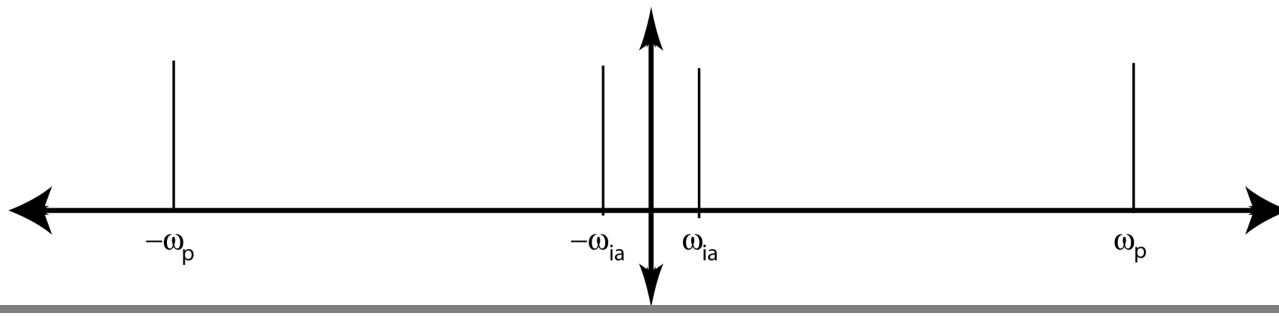
Damped resonance

- Waves in a plasma are resonances.
- Damped resonances are not sharp
  - Example – Q of a resonant circuit.
- IS: Thermal ions have motions close to ion-acoustic speed (Landau damping – “surfing”; locked to I-A waves)

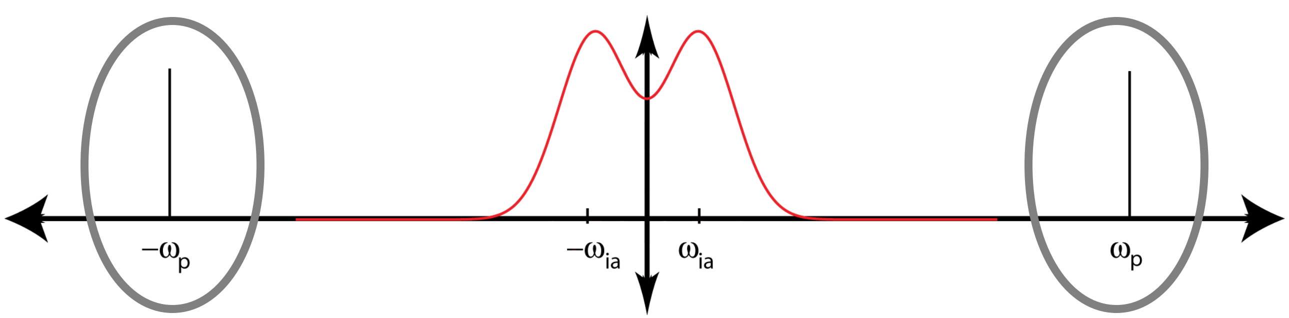


Resonance　　　　Damped Resonance

Wave Spectrum (ISR Spectrum)



Why aren't the Langmuir (plasma) waves damped?
Electron thermal velocity ~ 125 km/s but plasma wave frequency ~ several MHz – Not much interaction and not much damping.



Nyquist Theorem

Use electron force/response concept and solve for electron and ion admittances y_e, y_i (analogous to resistive dissipation). Arrive at spectral expression

$$\sigma_0(\omega_o + \omega)d\omega = N_0 r_e^2 \,\mathrm{Re}\left\{ \frac{y_e(y_i + jk^2\lambda_{de}^2)}{y_e + y_i + jk^2\lambda_{de}^2} \frac{d\omega}{\pi\omega} \right\}$$

- Short wavelength limit ($k^2\lambda_{de}^2 >> 1$): pure e^- scatter
- Long wavelength limit: RHS $\longrightarrow y_e y_i/(y_e + y_i)$: damped ion-acoustic resonances
- Near plasma frequency: $y_e + y_i + jk^2\lambda_{de}^2 \longrightarrow 0$: plasma lines

Incoherent Scatter Spectral Dependence

Spectral response can be evaluated using these frameworks for:

- Thermal inequality $T_e \neq T_i$: decreases Landau damping
- Ion-neutral collisions ν_{in}: narrows spectrum
- Background magnetic field B_0: makes electrons heavier

$$m_e \longrightarrow m_e^* = \frac{m_e}{\cos^2 \alpha}$$

Also, ion gyro-resonance (mass-dependent).

Incoherent Scatter Spectral Dependence

- Ion mixtures: $\frac{T_e}{T_i} y_i \rightarrow \sum_j \frac{T_e}{T_j} \frac{N_j}{N_0} y_j(m_j, T_j)$
- Unequal ion temperatures
- Particle drifts: $\omega \rightarrow \omega - \vec{k} \cdot \vec{v}_{de}$
- Plasma line measurements $([e^-], T_e, v_\parallel)$
- Photoelectron heating, non-Maxwellian plasmas
- Faraday rotation effects (equator, low TX freq)

Things can get hairy. For example, magnetic field evaluation requires Gordeyev integral:

$$\int e^{j(\theta - j\phi)t - \frac{\sin^2 \alpha}{\phi^2} \sin^2(\frac{\phi t}{2}) - \frac{t^2}{4} \cos^2 \alpha} dt$$

Measurement Statistics

$E_s(t)$ and $\therefore V_s(t)$ are Gaussian random variables (Central Limit Theorem):

$$
\begin{aligned}
V_s(t) = V_1 \quad &= x_1 + jx_2 \\
V_s(t+\tau) = V_2 \quad &= x_3 + jx_4
\end{aligned}
$$

We desire ensemble averages of 2nd moments (correlations):

$$\langle V_1 V_2^* \rangle = \langle (x_1 + jx_2)(x_3 + jx_4)^* \rangle = S\rho(\tau)$$

where S is signal power, and IS theory gives medium correlation

$$\rho(\tau) = \rho_R(\tau) + j\rho_I(\tau)$$

Measurement Statistics

In general, we define an estimator to approximate true ensemble average - e.g.

$$\hat{S} = \frac{1}{K}\sum_{i=1}^{K} V_i V_i^*$$

might be power estimator for true $S = \langle V_1 V_1^* \rangle$. Each estimator will have an associated bias and variance, e.g.

$$\text{bias} = \left\langle \hat{S} \right\rangle$$

$$\text{variance}(\hat{S} - S) = \left\langle (\hat{S} - S)^2 \right\rangle$$

Power Estimation

For total scattered power, use

$$\hat{S} = \frac{1}{K}\sum_{i=1}^{K} V_i V_i^*$$

$$\text{Bias}: \ \hat{S} = S$$

$$\text{Variance}: \ \left\langle (\hat{S} - S)^2 \right\rangle = \frac{S^2}{K} = \delta_S^2$$

$$\text{RMS frac error}: \ \frac{\delta_S}{S} = \frac{1}{\sqrt{K}}$$

10,000 samples needed for 1% accuracy.

Power Estimation: Noise Effects

Add noise (Gaussian RV with different 2nd moment).
Use estimator

$$\hat{S} = S \hat{+} N - \hat{N}$$

$$\text{Bias}: \ \hat{S} = S$$

$$\text{Variance}: \ \delta_S^2 \sim \frac{(S+N)^2}{K_{S+N}}$$

$$\text{RMS frac error}: \ \frac{\delta_S}{S} \sim \frac{S+N}{S} \frac{1}{\sqrt{K_{S+N}}}$$

ACF Estimation

We want $\langle V(t)V^*(t+\tau)\rangle = \langle V_1 V_2^*\rangle = S\rho(\tau)$. A popular estimator is:

$$\hat{\rho} = \frac{\frac{1}{K}\sum_{i=1}^{K} V_{1i}V_{2i}^*}{\left[\frac{1}{K^2}\sum_{i=1}^{K}|V_{1i}^2|\sum_{i=1}^{K}|V_{1i}^2|\right]^{\frac{1}{2}}} = \frac{A}{B}$$

After linearizing and lots of details:

$$\text{Bias}: \ \hat{\rho} = \rho(1 - \frac{1}{4K}(1 - |\rho|^2))$$

$$\text{Variance}: \ \delta_\rho^2 = \frac{1}{K}\left[1 - \frac{3}{2}|\rho|^2 + \frac{1}{2}|\rho|^4\right]$$

ACF Estimation: Noise Effects

Effect of adding noise is to change the estimator:

$$\hat{\rho} = \frac{A_{S+N} - A_N}{B_{S+N} - B_N}$$

Details show that

$$\delta_{\hat{\rho}}^2 \sim \frac{1}{K} \left(\frac{S+N}{S} \right)^2 \left[1 + \frac{1}{2} |\rho_S|^2 \right]$$

Consequences:

- When SNR low, variance large
- Larger S is wasted statistically